Math 131B: Analysis – Homework 1

Due: April 12, 2017

Exercise 1 (From Tao).

Let $n \in \mathbb{N}$ and let $a_i, b_i \in \mathbb{R}$ $i \in \{1, \ldots, n\}$. Show the identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

Using the previous identity show the *Cauchy-Schwaz inequality*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}} \tag{C-S}$$

Using (C-S) prove the *triangle inequality*

$$\left(\sum_{i=1}^{n}(a_i - b_i)^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}$$

Exercise 2 (From Tao).

(1) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and let $x \in \mathbb{R}$. Show that $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if $\lim_{n\to+\infty} d(x_n, x) = 0$, where $d(a, b) := |a - b|, \quad \forall a, b \in \mathbb{R}$.

(2) Show that d_{l^1}, d_{l^2}, d_{l^∞} and d_{disc} introduced during the first lecture define metrics on $X = \mathbb{R}^n$, $n \in \mathbb{N}$. Adjusting the definitions of d_{l^1}, d_{l^2} and d_{l^∞}, show that they define metrics on $Y := \left\{(a_n)_{n\in\mathbb{N}} : \sum_{n\ge1} |a_n| < +\infty\right\}$, the space of absolutely summable sequences.

(3) Let $X = [0, 1]$ and $f : X \to X$ be defined as $f(x) = x$ if $x \in (0, 1)$, $f(0) = 1$ and $f(1) = 0$. Let $d(x, y) := |f(x) - f(y)|$ for all $x, y \in X$. Show that d defines a metric on X.

Exercise 3.

Prove the following inequalities for d_{l^1}, d_{l^2} and d_{l^∞} on $\mathbb{R}^n$:

$$d_{l^2}(x, y) \le d_{l^1}(x, y) \le \sqrt{n}\, d_{l^2}(x, y), \ \forall x, y \in \mathbb{R}^n;$$

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \le d_{l^\infty}(x, y) \le d_{l^2}(x, y), \ \forall x, y \in \mathbb{R}^n$$

and deduce the equivalence of d_{l^1} and d_{l^∞} on $\mathbb{R}^n$.

Hint: use (C-S) when it is needed.

Exercise 4 (From Tao).

(1) Let (X, d) be a metric space, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ two sequences in X and $x, y \in X$. Suppose that $(x_n)_{n\in\mathbb{N}}$ converges w.r.t. d to x and $(y_n)_{n\in\mathbb{N}}$ converges w.r.t. d to y. Show that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Hint: use the triangle inequality several times.

(2) Draw the unit balls in $\mathbb{R}^2$ w.r.t. the metrics d_{l^1}, d_{l^2}, d_{l^∞} and d_{disc}. Explain/sketch the unit ball w.r.t. d_{l^3}.