Math 131B: Analysis – Homework 2

Due: April 19, 2017

Exercise 1 (From Tao)**.**

Let (X, d) be an arbitrary metric space and $E \subseteq X$. Prove the following statements:

(1) E if open (i.e. $E \cap \partial E = \emptyset$) if and only if $E = \mathrm{int}(E)$.

(2) Any singleton $\{x_0\}$, where $x_0 \in X$ is a closed set.

(3) E is closed if and only if $X \setminus E$ is open.

(4) $\overline{E}$ is the smallest closed set that contains E. More precisely, if $K \subseteq X$ is closed and $E \subseteq K$, then $\overline{E} \subseteq K$.

(5) $E \subseteq \overline{E}$.

Exercise 2 (From Tao)**.**

Let (X, d) be a metric space, let $x_0 \in X$, $r > 0$ and let us consider the open ball $B := \{x \in X : d(x_0, x) < r\}$ and the closed ball $C := \{x \in X : d(x_0, x) \leq r\}$. Show that

(1) $\overline{B} \subseteq C$. *Hint:* use the previous exercise.

(2) Give an example of (X, d), $x_0 \in X$ and $r > 0$ for which $\overline{B} \neq C$.

Hint: you can use any property proven during the lectures.

Exercise 3.

(1) Let (X, d) be a metric space and $Y \subset X$. Show that $d_Y : Y \times Y \to [0, +\infty)$, i.e. $d_Y = d|_{Y \times Y}$ the restriction of d to the set $Y \times Y$ defines a metric on Y.

(2) Let $X = [0, 1] \subset \mathbb{R}$ and $Y = \{1/n : n \in \mathbb{N}\} \subset X$. Let $d(x, y) = |x - y|$, $\forall x, y \in X$ the usual metric on X and d_Y the induced metric on Y. Show that $E = \left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$ is relatively open in Y and it is not open in X. What are $\overline{E}^X$ and $\overline{E}^Y$, i.e. the closure of E in (X, d) and in (Y, d_Y) respectively?

Exercise 4 (From Tao)**.**

Let (X, d) be a metric space and E, Y two sets such that $E \subset Y \subset X$. Show that E is relatively closed in Y if and only if there exists $K \subseteq X$ a closed set in (X, d) such that $E = K \cap Y$.

Hint: for the construction of K try to look at the closure of E in (X, d).

Exercise 5 (based on Tao)**.**

Let (X, d) be an arbitrary metric space. Prove the following statements:

(1) If a sequence is convergent in X, all its subsequences are converging to the same limit as the original sequence.

(2) If a subsequence of a Cauchy sequence is convergent, then the whole sequence is convergent to the same limit as the subsequence.

(3) Suppose that (X, d) is complete and $Y \subseteq X$ is closed in (X, d). Then the space $(Y, d|_{Y \times Y})$ is complete.

(4) Give two examples of metric spaces (with explanations), where every subset of the original space is bounded.