Math 131B: Analysis – Homework 3

Due: April 26, 2017

Exercise 1 (based on Tao)**.**

(1) Let (X, d) be a metric space and let $\{K_i\}_{i\in\mathbb{N}}$ be a family of non-empty compact subsets in (X, d) such that

$$K_1 \supset K_2 \supset \dots.$$

Show that $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$.

Hint: in the metric space $(K_1, d|_{K_1\times K_1})$ consider the sets $V_n := K_1 \setminus K_n$ for all $n \geq 1$ and show that these are open. Then suppose that the intersection is empty, use some De Morgan laws and use the 'open cover' characterization of compact sets to get a contradiction.

(2) Give and example of a metric space (X, d), a closed subset of it $E \subset X$ and an open cover of this subset, i.e. $E \subseteq \bigcup_{\alpha\in I} V_\alpha$ where $\{V_\alpha\}_{\alpha\in I}$ is a family of open sets, I is infinite and there exists no $F \subset I$ finite such that $E \subseteq \bigcup_{\alpha\in F} V_\alpha$.

Hint: think to the discrete metric.

Exercise 2.

(1) Let (X, d) be a metric space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x_0 \in X$. Prove hat the following two statements are equivalent.

(a) For any $\varepsilon > 0$, $B(x_0, \varepsilon)$ contains infinitely many elements of the sequence $(x_n)_{n\in\mathbb{N}}$.

(b) There exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ that is converging to x_0 w.r.t. d.

(2) Prove the "converse implication" of the open-cover characterization of compactness. More precisely let (X, d) be a metric space and $K \subseteq X$ such that for any open family $\{V_\alpha\}_{\alpha\in I}$ that covers K ($K \subseteq \cup_{\alpha\in I} V_\alpha$) there exists $J \subseteq I$ a finite set such that $K \subseteq \cup_{\alpha\in J} V_\alpha$. Then K is (sequentially) compact.

Hint: first suppose that K is not a finite set (otherwise it should be trivial). Then work by contradiction, and use the fact that $(1)(a)$ fails for a sequence. Choose a special open cover of K then a finite sub-cover and contradict the fact that the sequence has infinitely many elements.

Exercise 3 (From Tao)**.**

Let (X, d) be a metric spec.

(a) If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact if and only if Z is closed.

(b) If $Y_1, \dots, Y_n$ is a finite collection of compact subsets of X, then their union $Y_1 \cup \dots \cup Y_n$ is also compact.

(c) Every finite subset of X (including the empty set) is compact.

Exercise 4 (from Tao)**.**

(1) Let (X, d_{disc}) an arbitrary metric space with the discrete metric. Show that X is always complete.

(2) In this last situation, when is X compact and when it is not? Prove your claim.

Exercise 5 (from Tao)**.**

Let (X, d) be a metric space, $E \neq \emptyset$ a compact subset of X, and let $x_0 \in X$. Show that there exists $x \in E$ such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e. x is the closest point to x_0 from E.

Hint: let R be the above infimum and construct a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $d(x_0, x_n) \leq R+1/n$ and then use the compactness of E.