Math 131B-1: Analysis – Homework 5

Due: May 10, 2016

Exercise 1 (From Tao)**.**

(1) Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two uniformly continuous functions. Show that $g \circ f : X \to Z$ is also uniformly continuous.

(2) Let X be a subset of the real line $\mathbb{R}$. Prove that the following statements are equivalent.

  (a) X is connected.

  (b) X is an interval.

(3) Show that a metric space (X, d) is disconnected if and only if X contains a proper subset which is simultaneously open and closed.

Exercise 2 (From Tao)**.**

(1) Exercises 2.1.4 and 2.1.5 from Tao (page 40).

(2) Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. (*Hint:* use the previous exercise.)

(3) Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha \in I}$ be a collection of connected sets in X. Suppose also that $\cap_{\alpha \in I} E_\alpha$ is non-empty. Show that $\cup_{\alpha \in I} E_\alpha$ is connected.

Exercise 3 (From Tao)**.**

(1) Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$. there exists a continuous function $\gamma : [0, 1] \to E$ from the unit interval $[0, 1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every path-connected set is connected.

(2) Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

Exercise 4 (From Tao)**.**

(1) Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f : E \to Y$ be a function, and let $x_0 \in E$. Show that the limit $\lim_{x \to x_0; x \in E} f(x)$ exists if and only if the limit $\lim_{x \to x_0; x \in E \setminus \{x_0\}} f(x)$ exists and is equal to $f(x_0)$. Also, show that if the limit $\lim_{x \to x_0; x \in E} f(x)$ exists at all, then it must be equal to $f(x_0)$.

(2) Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $x_0 \in X$, $y_0 \in Y$ and $z_0 \in Z$. Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let E be a set. If we have $\lim_{x \to x_0; x \in E} f(x) = y_0$ and $\lim_{y \to y_0; x \in f(E)} g(y) = z_0$, conclude that $\lim_{x \to x_0; x \in E} (g \circ f)(x) = z_0$.

(3) Show that in Proposition 3.1.5. (a) is equivalent to (d).