Math 131B-1: Analysis – Homework 6

Due: May 17, 2016

Exercise 1 (From Tao)**.**

(a) Exercise 3.2.1 and 3.2.4. from Tao (page 52-53 in my version).

(b) Let (X, d_X) and (Y, d_Y) be two metric spaces and show that the space of bounded functions $B(X;Y)$ with the metric $d_\infty(f, g) = \sup_{x\in X} d_Y(f(x, g(x)))$ is a metric space.

Exercise 2 (From Tao)**.**

(1) Exercise 3.3.8 from Tao (page 55).

(2) Show that the statement of Proposition 3.3.3 fails, if (Y, d_Y) is not complete. *Hint:* construct a counterexample.

Exercise 3 (From Tao)**.**

(1) Exercise 3.7.1 from Tao (page 66).

(2) Prove Theorem 3.7.1 from Tao without assuming that the sequence f_n' is integrable[1]. For this you can use the hint given in the textbook, write the MVT for $f_n - f_m$. Mind that you have to show also that the (uniform) limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ is differentiable with derivative g.

Exercise 4.

Construct a sequence of Riemann integrable functions $(f_n)_{n\in\mathbb{N}}$ on a subset E of $\mathbb{R}$ which is converging pointwisely to a Riemann integrable function $f : E \to \mathbb{R}$ such that

$$\int_E f(x)\mathrm{d}x = \lim_{n\to+\infty} \int_E f_n(x)\mathrm{d}x,$$

but $(f_n)_{n\in\mathbb{N}}$ is not converging uniformly to f.

[1] For those of you who are interested, there are differentiable functions with bounded derivatives, that are not Riemann integrable, see for instance the example constructed by Volterra, https://en.wikipedia.org/wiki/Volterra's_function. But the analysis of this example requires some non-elementary tools, which were missing from Math 131A.