Math 131B-1: Analysis – Homework 7

Due: May 24, 2017

Exercise 1 (About the Arzèla-Ascoli theorem).

(1) Show that the assumptions in the Arzèla-Ascoli theorem are also necessary, meaning that if one has a compact family of continuous function in $(C([a,b];\mathbb{R}); d_\infty)$, this family is uniformly bounded and equicontinuous.

(2) Show that the subsequence that we have chosen via the diagonalization argument is indeed convergent at every rational from $[a,b]$.

(3) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ on $[0,1]$ defined as $f_n(x) = x^n$ does not have a convergent subsequence w.r.t. the uniform convergence. To show this, by (1) it is enough to show that it is not equicontinuous.

Exercise 2 (Example 3.7.4 and Exercise 4.7.10 from Tao).

(1) Use Weierstrass' M-test to show that the series $\sum_{n=1}^{+\infty} 4^{-n}\cos(32^n \pi x)$ is uniformly convergent to a continuous function on the whole $\mathbb{R}$.

(2) Show that the limit function f is not differentiable at any point $x_0 \in \mathbb{R}$. To do so, use the hints from Exercise 4.7.10. Basically you have to construct a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ that is converging to x_0 as $n \to +\infty$ but the absolute value of the differential quotient $\left|\frac{f(x_n)-f(x_0)}{x_n - x_0}\right|$ is diverging to $+\infty$, and this proves the non-differentiability of f at x_0.

Exercise 3.

Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$, $f_n : [0,2] \to \mathbb{R}$, defined as $f_n(x) = \frac{\ln(1+nx^2)}{2n}$ is converging uniformly to a differentiable function. Why can we interchange the differentiation and the limit as $n \to +\infty$? Determine the limit function.

Hint: study the uniform convergence of the sequence $(f_n')_{n\in\mathbb{N}}$ and use the theorem that we proved on uniform convergence and differentiation.

Exercise 4.

Show that the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}\cos(4nx)$ is uniformly convergent to a continuous function f on the whole $\mathbb{R}$ and show that $\int_0^{\pi/2} f(x)\,\mathrm{d}x = 0$.