Math 32B-2: Calculus of Several Variables – Homework #4

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Exercise 1 (From Rogawski-Adams).

(1) Find a mapping G that maps the disk $u^2 + v^2 \leq 1$ onto the interior of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$. Then use the change of variables formula to prove that the area of the ellipse is πab.

(2) Compute the area of the region enclosed by the ellipse $x^2 + 2xy + 2y^2 - 4y = 8$ as an integral in the variables $u = x + y$, $v = y - 2$.

(3) Use the change of variables formula in three variables to prove that the volume of the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ is equal to $abc \times$ the volume of the unit sphere.

Exercise 2 (From Rogawski-Adams).

Sketch the region D and integrate $f(x, y)$ over D using polar coordinates.

(1) $f(x, y) = y(x^2 + y^2)^3$; $y \geq 0$, $x^2 + y^2 \leq 1$.

(2) $f(x, y) = e^{x^2+y^2}$; $x^2 + y^2 \leq R$, $R > 0$ is given.

(3) $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{x^2 + y^2}\, \mathrm{d}x\, \mathrm{d}y$

(4) $\displaystyle\int_0^2 \int_x^{\sqrt{3}x} y\, \mathrm{d}y\, \mathrm{d}x$.

Exercise 3 (From Rogawski-Adams).

(1) Let W be the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$.

   (a) Show that the projection W on the xy-plane is the disk $x^2 + y^2 \leq 2$.

   (b) Compute the volume of W using polar coordinates.

(2) Use cylindrical coordinates to calculate the integral of the function $f(x, y, z) = z$ over the region above the disk $x^2 + y^2 \leq 1$ in the xy-plane and below the surface $z = 4 + x^2 + y^2$.

(3) Use cylindrical coordinates to calculate $\displaystyle\int\int\int_W f(x, y, z)\, \mathrm{d}V$ for $f(x, y, z) = xz$, where

$$x^2 + y^2 \leq 1,\ x \geq 0,\ 0 \leq z \leq 2.$$

(4) Use spherical coordinates to calculate the triple integral of $f(x, y, z) = x^2 + y^2 + z^2$ over the region

$$1 \leq x^2 + y^2 + z^2 \leq 4.$$

Exercise 4 (From Rogawski-Adams).

(1) Use cylindrical coordinates to find the volume of a sphere of radius a from which a central cylinder of radius b has been removed, where $0 \leq b \leq a$.

(2) Use cylindrical coordinates to find the volume of the region bounded below by the plane $z = 1$ and above by the sphere $x^2 + y^2 + z^2 = 4$.