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Abstract

In this thesis we collected some recent results from the theory of Picard operators and its applications. This method was introduced by Prof. Ioan A. Rus and used later on by many mathematicians (see the references). It seemed to be very powerful tool in the theory of ordinary- and partial differential equations (to study existence, uniqueness, stability and differentiability of the solutions, etc.), integral equations and inequalities and also in many other type of problems related to fixed point theory.

Most of the results contained in the present thesis are taken from the recent papers [8], [9] and [10], and some of them were presented as well at the *4th International Conference on Nonlinear Operators, Differential Equations and Applications, ICNODEA 2011*, Cluj-Napoca, Romania, July 5-8, 2011 by the author and also by his advisor.

After a short introduction of Picard operators, some abstract Gronwall lemmas, Ulam-Hyers stability and the relation between them, in the following chapters we will discuss in a more detailed way some specific problems, applications of these results.

In Chapter 1 we present some new improvements of the well known Wendroff inequality and give the representation for the best estimation. We also prove some nonlinear Bihari-Wendroff type inequalities. These results are contained in [8] and were motivated by [1].

In Chapter 2 we extend the results obtained in the previous chapter to arbitrary time scales. Moreover we study some nonlinear integral inequalities on time scales. These results (published in [9]) improve some recently obtained results in [19], [4] and [5].

In Chapter 3 we continue our work with the study of the Ulam-Hyers stability of dynamical equations on time scales. We present some results for bounded time scale intervals and for unbounded ones as well. In the case of unbounded domains we cannot use the usual operatorial framework of Picard operators, but using some direct methods we could have obtained some positive results, which are more general, than the ones discussed in [6]. And due to the fact, that we are working on time scales, these results will unify the stability properties of differential and difference equations (see [31],[32],[16] and [24]-[26]). The chapter is mainly containing the results from [10].

In Chapter 4 we will use the same ideas to study the Ulam-Hyers stability of some linear and nonlinear elliptic PDEs on bounded, connected domains with Lipschitz boundary. In the case of this chapter the main tools, which are used are not necessarily the abstract techniques of the Picard operators, but there is a big accent on the Sobolev embeddings (due to the bounded domains), Poincaré's and the Cauchy-Schwartz inequalities.

Finally in Chapter 5 we will present some further research possibilities about improving some stochastic Wendroff type inequalities, studying the Ulam-Hyers stability of stochastic ODEs and PDEs, and some other type of (evolution) PDEs.

This work is the result of my own activity. I have neither given nor received unauthorized assistance on this work.

Contents

Introduction

0.1 Preliminaries on Picard operators

The theory of Picard operators was introduced by Prof. Ioan A. Rus (see [34], [35], [36] and their references) to study problems related to fixed point theory. This abstract approach was used later on by many mathematicians and it seemed to be a very useful and powerful method in the study (beside many others) of integral equations and inequalities, ordinary- and partial differential equations (existence, uniqueness, differentiability of the solutions), etc.

We recommend the textbooks [7],[36] and the references therein for a deep insight to this theory and its applications. In what follows we will give some basic definitions and results, mainly following the notations from [34], [35] and [36].

Let $(X, \to)$ be an L-space (i.e. a nonempty set X with a convergence structure, see [35]), $A : X \to X$ an operator. We denote by F_A the set of the fixed points of A. We also denote $A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A^n \circ A,\ n \in \mathbb{N}$ the iterate operators of the operator A.

Definition 0.1.1 ([38])**.** By definition $A : X \to X$ is weakly Picard operator if the sequence of successive approximations, $A^n(x)$, converges for all $x \in X$ and the limit (which may be depend on x) is a fixed point of A.

Definition 0.1.2 ([34], [35], [36])**.** A is a Picard operator (briefly PO), if there exists $x_A^* \in X$ such that:

(i) $F_A = \{x_A^*\}$;

(ii) $A^n(x) \to x_A^*$ as $n \to \infty, \forall x \in X$.

Equivalently we can say, that if for a weakly Picard operator $A : X \to X$ $F_A = \{x_A^*\}$, then A is a PO.

The following class of weakly Picard operators is very important in our consideration. Let (X, d) be a metric space.

Definition 0.1.3 ([38])**.** Let $A : X \to X$ be a weakly Picard operator and $c > 0$ a real number. By definition the operator A is c-weakly Picard operator if

$$d(x, A^\infty(x)) \leq c \cdot d(x, A(x)),\ \forall x \in X.$$

We present two examples for c-weakly Picard operators from [38].

Example 0.1.4. Let (X, d) be a complete metric space and $A : X \to X$ an operator with closed graphic. We suppose that A is graphic α-contraction, i.e.

$$d(A^2(x), A(x)) \leq \alpha \cdot d(x, A(x)),\ \forall x \in X.$$

Then A is a c-weakly Picard operator, with $c = (1 - \alpha)^{-1}$.

Example 0.1.5. Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ a function and $A : X \to X$ an operator with closed graphic. We suppose that:

(i) A is a φ-Caristi operator, i.e.

$$d(x, A(x)) \leq \varphi(x) - \varphi(A(x)),\ \forall x \in X;$$

(ii) there exists $c > 0$ such that

$$\varphi(x) \leq c \cdot d(x, A(x)),\ \forall x \in X.$$

Then A is a c-weakly Picard operator.

Now let us say a few words about Ulam-Hyers stability in the theory of functional equations and its generalization in metric spaces.

0.2 Stability problems studied by Ulam, Hyers and Rassias

In 1940, S.M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin, in which discussed a number of important unsolved problems. These problems were also discussed in [43]. Among those was the question concerning the stability of group homomorphisms, namely:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,\ \forall x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon,\ \forall x \in G_1?$$

The case of approximately additive functions was solved in the next year by D.H. Hyers ([21]) under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality

$$||f(x + y) - f(x) - f(y)|| \leq \varepsilon,\ \forall x, y \in G_1,$$

can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x + y) = f(x) + f(y)$, is said to have the Ulam-Hyers stability.

Furthermore the result of Hyers has been generalized by Th. M. Rassias in 1978 ([33]), he attempted to weaken the condition for the bound of the Cauchy difference as follows:

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p),$$

and proved the Hyers theorem. That is, Rassias proved the Ulam-Hyers-Rassias stability of the Cauchy additive functional equation.

Since then, the stability of many functional equations have been extensively investigated, and also it is a very important field of research in the theory of stability of functional equations (see [23]-[29],[31],[38]).

In the near past many research papers have been published in this field of Mathematics, but we can say that in most of them there were used some special direct methods, constructions, which in many other cases could be used only with difficulties. The uniform approach with Picard operators to the discuss of the stability problems of Ulam-Hyers type is due to I. A. Rus (see [38]).

Now we give the definition of Ulam-Hyers stability of a fixed point equation due to I. A. Rus.

By the analogy with the notion of Ulam-Hyers stability in the theory of functional equations we have

Definition 0.2.1 ([38])**.** Let (X, d) be a metric space and $A : X \to X$ be an operator. By definition, the fixed point equation

$$x = A(x) \tag{0.2.1}$$

is said to be Ulam-Hyers stable if there exists a real number $c_A > 0$ such that: for each $\varepsilon > 0$ real number and each solution y^* of the inequality

$$d(y, A(y)) \leq \varepsilon,$$

there exists a solution x^* of the equation (0.2.1) such that

$$d(y^*, x^*) \leq c_A \cdot \varepsilon.$$

Definition 0.2.2 ([40])**.** The equation

$$x'(t) = f(t, x(t)), \, \forall t \in [a, b) \tag{0.2.2}$$

is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a, b))$ of the inequality

$$|y'(t) - f(t, y(t))| < \varepsilon, \;\; \forall t \in [a, b)$$

there exists a solution $x \in C^1([a, b))$ of the equation 0.2.2 with the property

$$|y(t) - x(t)| \leq c_f \varepsilon, \forall t \in [a, b).$$

Now we present a very important characterization theorem from the point of view of Ulam-Hyers stability:

Theorem 0.2.3 ([38]-Remark 2.1)**.** *Let (X,d) be a metric space. If $A : X \to X$ is a c-weakly Picard operator, then the fixed point equation* (0.2.1) *is Ulam-Hyers stable.*

Remark 0.2.4. We can conclude, that the theory of Picard operators is a very powerful tool in the study of Ulam-Hyers stability of functional equations. We only have to define a fixed point equation from the functional equation we want to study, then if the defined operator is c-weakly Picard we also have immediately the Ulam-Hyers stability of the desired equation.

Of course it is not always possible to transform a functional equation or a differential equation into a fixed point problem (for example we need some assumptions on the domain of the ODEs, PDEs, etc.) and actually this point shows a weakness of this theory.

Lemma 0.2.5 ([38])**.** *Let (X, d) be a Banach space. If an operator $A : X \to X$ is a contraction with the positive constant $q < 1$, then A is c-weakly Picard operator with the positive constant $c_A = \frac{1}{1-q}$. Moreover the fixed point equation* (0.2.1) *is Ulam-Hyers stable.*

0.3 Abstract Gronwall lemmas

In the following we will present some abstract Gronwall type lemmas, as applications of the Picard operators.

Lemma 0.3.1 ([34], [35])**.** *(Abstract Gronwall lemma) Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We assume that:*

(i) A is PO;

(ii) A is increasing.

If we denote by x_A^* the unique fixed point of A, then:

(a) $x \leq A(x) \Rightarrow x \leq x_A^*$;

(b) $x \geq A(x) \Rightarrow x \geq x_A^*$.

Lemma 0.3.2 ([34], [35])**.** *(Abstract Gronwall-comparison lemma) Let $(X, \to, \leq)$ be an ordered L-space and $A_1, A_2 : X \to X$ be two operators. We assume that:*

(i) A_1 is increasing;

(ii) A_1 and A_2 are POs;

(iii) $A_1 \leq A_2$.

If we denote by x_2^* the unique fixed point of A_2, then

$$x \leq A_1(x) \Rightarrow x \leq x_2^*.$$

These lemmas are very powerful because once we prove that the operator is a Picard operator and we have an L-space structure, the Gronwall type inequalities can be proved without any additional effort (calculation). In many Gronwall type inequalities the upper bound of the solution is the solution of the corresponding fixed point equation. These can be proved using Lemma 0.3.1. This is not the case of the Wendroff type inequalities, where the upper bound is not the solution of the corresponding fixed point equation (see [17]). To handle these cases Lemma 0.3.2 can be used (see [41]). The main difficulty in using Lemma 0.3.2 is the construction of the operator A_2. To avoid this we proposed the following variant in [9]:

Lemma 0.3.3. *([9] - Abstract Gronwall lemma) Let* $(X, \to, \leq)$ *be an orderd L-space and* $A : X \to X$ *be an operator with the following properties:*

(i) A *is increasing;*

(ii) A *is PO;*

(iii) there exists $\overline{x}$ *with the property* $A\overline{x} \leq \overline{x}$.

If for some $x \in X$ *we have* $x \leq Ax$*, then* $x \leq \overline{x}$.

Proof. A is increasing, so the inequality $x \leq Ax$ implies $x \leq A^n x$, $\forall n \in \mathbb{N}$. Due to the Picard property of the operator A this implies $x \leq x^*$, where x^* is the unique solution of the operator A. On the other hand the inequality $A\overline{x} \leq \overline{x}$ implies $A^n\overline{x} \leq \overline{x}$ and so $x^* \leq \overline{x}$, which completes the proof. □

Remark 0.3.4. If the conditions of Lemma 0.3.1 or 0.3.2 are satisfied, than the conditions of Lemma 0.3.3 are also satisfied. From this viewpoint Lemma 0.3.3 is more general than Lemma 0.3.1 and Lemma 0.3.2. We have to mention that in many cases the inequality $A\overline{x} \leq \overline{x}$ can be established by using the operator A_2 with the properties $A \leq A_2$ and $A_2\overline{x} = \overline{x}$. Our result from [9] can not be proved with this technique because the operator A_2 for which $A_2\overline{x} = \overline{x}$ does not satisfy $A \leq A_2$. This motivates the necessity of Lemma 0.3.3.

Basically these are the key definitions and results with the help of which we will build the content of the present thesis. Later on we will introduce some notions in the case of the study of these kind of problems on time scales and further in the case of elliptic PDEs.

Chapter 1

Wendroff type inequalities

The Wendroff inequality is a generalization of the well-known Gronwall inequality for 2 independent variables, has its origin in the theory of partial differential equations and can be found in many monographs on inequalities ([11], [12],[28], [30]). Recently the authors in [1] gave a sharpened version for a Wendroff type inequality proved by Pachpatte (see [30]) but unfortunately their proof contains some errors.

In [8] we proved the inequality given in [1] (Theorem 2.2) and we used the abstract comparison Gronwall lemma to obtain new proofs for well known generalizations of the Wendroff inequality. Our method uses an operatorial point of view and can be used to simplify the proofs of many other Wendroff type inequalities.

In what follows, we present the results obtained in [8].

1.1 The inequality and its improvements in [1]

We consider $D = [0, l] \times [0, l] \subset \mathbb{R}^2$. As a starting point we recall the following generalization of the original Wendroff inequality proved by B. G. Pachpatte:

Theorem 1.1.1 ([1],[30]). *Let $u(x, y)$, $w(x, y)$ and $a(x, y)$ non-negative continuous functions defined for $(x, y) \in D$, and let $w(x, y)$ be non-decreasing in each variable $x, y \in D$. If*

$$u(x, y) \leq w(x, y) + \int_0^x \int_0^y a(s, t)u(s, t)dtds,\ \forall (x, y) \in D \tag{1.1.1}$$

then

$$u(x, y) \leq w(x, y) \exp\left(\int_0^x \int_0^y a(s, t)dtds\right),\ \forall (x, y) \in D. \tag{1.1.2}$$

This theorem was generalized in [1] as follows:

Theorem 1.1.2 ([1] - Theorem 2.1). *Suppose $u(x, y)$, $w(x, y)$ and $a(x, y)$ are non-negative continuous functions defined on a domain D. If inequality* (1.1.1) *is satisfied for all $(x, y) \in D$, then*

$$u(x, y) \leq w(x, y) + G^{-1}\left(\int_0^x \int_0^y a(s, t)dtds\right),\ \forall (x, y) \in D, \tag{1.1.3}$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{s+w}, r \geq r_0 > 0, \tag{1.1.4}$$

G^{-1} *is the inverse function of* G *and* $\int_0^x \int_0^y a(s,t)dtds \in Dom(G^{-1}),\ \forall (x,y) \in D.$

Theorem 1.1.3 ([1] - Theorem 2.2)**.** *Suppose* $u(x,y)$*,* $w(x,y)$ *and* $a(x,y)$ *are non-negative continuous functions defined on a domain* D*, and let* $w(x,y)$ *be nondecreasing in each variable* $(x,y) \in D$*. If* u *satisfies inequality* (1.1.1)*, then*

$$u(x,y) \leq w(x,y) \left[1 + \int_0^x \int_0^y a(s,y) \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\xi d\eta\right) dtds\right],\ \forall (x,y) \in D. \tag{1.1.5}$$

Remark 1.1.4. The proof of Theorem 2.2 in [1] contains 2 errors. The first error is on line 5-6 of the proof and can be corrected only by adding further assumptions on the functions w and a. This motivates the need of a new proof for this theorem. The second error is on page 611, line 11 but this error can be corrected only by replacing the right hand side of the inequality with an other expression and this weakens the inequality.

1.2 The representation of the best estimation

The inequation (1.1.1) is linear in u, so we can obtain a representation by applying the successive approximation method. This representation gives also the solution of the integral equation

$$u(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)u(s,t)dtds,\ \forall (x,y) \in D \tag{1.2.1}$$

starting from the $u_0 = w$, hence this is the maximal solution of the inequality (1.1.1). In [17] the authors proved that the right hand side of the classical Wendroff inequality is not the fixed point of the corresponding integral operator (it is not the solution of the associated integral equation). In what follows we present some results from [8], which say that this is also valid for the Wendroff type inequalities proved in [1] and we give the construction of the best possible estimation (see in [8]). We use this representation to give a correct proof of theorem 2.2 from [1].

Theorem 1.2.1 (Sz. András and A. Mészáros - [8])**.** *Suppose* $u(x,y)$*,* $w(x,y)$ *and* $a(x,y)$ *are non-negative continuous functions defined on a domain* D*. If* u *satisfies inequality* (1.1.1)*, then*

$$u(x,y) \leq w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)H(x,y,s,t)dtds,\ \forall (x,y) \in D \tag{1.2.2}$$

where

$$H(x,y,s,t) = \sum_{j=0}^{\infty} K_j(x,y,s,t) \text{ and}$$

$$K_{j+1}(x,y,s,t) = \int_s^x \int_t^y a(\xi,\eta)K_j(x,y,\xi,\eta)d\eta d\xi,\ K_0 \equiv 1.$$

Proof. The integral operator $A : C(D) \to C(D)$ defined by

$$A(u)(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)u(s,t)dtds \tag{1.2.3}$$

is a Picard operator if we consider the Bielecki norm on the set $C(D)$:

$$\|u\| = \max_{(x,y)\in D} e^{-\tau(x+y)}|u(x,y)|.$$

Moreover the space $(C(D), \|\cdot\|)$ is an ordered Banach space with the natural ordering $u \leq v \Leftrightarrow u(x,y) \leq v(x,y)$, $\forall (x,y) \in D$ and the operator A is an increasing operator. These observations allow us to apply the abstract Gronwall lemma, so

$$u(x,y) \leq u^*(x,y),$$

where $u^*(x,y)$ is the solution of the integral equation (1.2.1). But this solution can be obtained as the limit of the successive approximation sequence starting from $u_0 = w$ and the terms of this sequence can be calculated as follows:

$$\begin{aligned} u_1(x,y) &= A(u)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)dtds \end{aligned}$$

$$\begin{aligned} u_2(x,y) &= A(u_1)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)dtds + \\ &+ \int_0^x \int_0^y a(s,t) \int_0^s \int_0^t a(\xi,\eta)w(\xi,\eta)d\eta d\xi dtds \end{aligned}$$

Changing the order of integration in the last integral we obtain

$$\begin{aligned} u_2(x,y) &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)dtds + \\ &+ \int_0^x \int_0^y a(s,t)w(s,t) \int_s^x \int_t^y a(\xi,\eta)d\eta d\xi dtds \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t) \left[K_0(x,y,s,t) + K_1(x,y,s,t)\right] dtds. \end{aligned}$$

Applying the operator A one more time we obtain

$$\begin{aligned} u_3(x,y) &= A(u_2)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t) \sum_{j=0}^{2} K_j(x,y,s,t)dtds \end{aligned}$$

and by an inductive argument we deduce

$$\begin{aligned} u_{k+1}(x,y) &= A(u_k)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t) \sum_{j=0}^{k} K_j(x,y,s,t)dtds. \end{aligned}$$

Hence the solution can be represented as

$$u^*(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)H(x,y,s,t)dtds, \ \forall (x,y) \in D,$$

where

$$H(x,y,s,t) = \sum_{j=0}^{\infty} K_j(x,y,s,t) \text{ and}$$

$$K_{j+1}(x,y,s,t) = \int_s^x \int_t^y a(\xi,\eta)K_j(x,y,\xi,\eta)d\eta d\xi, \ K_0 \equiv 1.$$

□

Theorem 1.2.2 (Sz. András and A. Mészáros - [8] Corrected statement of theorem 2.2 from [1])**.** *Suppose* $u(x,y)$, $w(x,y)$ *and* $a(x,y)$ *are non-negative continuous functions defined on a domain* D *and* w *is nondecreasing in both variables. If* u *satisfies inequality* (1.1.1), *then*

$$u(x,y) \le w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t) \exp\left(\int_s^x \int_t^y a(\xi,\eta)d\eta d\xi\right) dtds, \ \forall (x,y) \in D. \tag{1.2.4}$$

Proof. Denote by $\overline{u}(x,y)$ the right hand side of the inequality (1.3.4). Using the representation from theorem 1.2.1 it is sufficient to prove that

$$H(x,y,s,t) \le \exp\left(\int_s^x \int_t^y a(\xi,\eta)d\eta d\xi\right), \ \forall (x,y) \in D.$$

In order to prove this inequality we proceed by mathematical induction and we prove that for all $k \in \mathbb{N}$

$$\sum_{j=0}^{k} K_j(x,y,s,t) \le \exp\left(\int_s^x \int_t^y a(\xi,\eta)d\eta d\xi\right), \ \forall (x,y) \in D \text{ and } s \le x, t \le y.$$

This inequality is trivial for $k = 0$. For a fixed k by replacing s with ξ and t with η, multiplying with $a(\xi,\eta)$ and integrating from s to x and from t to y we obtain

$$\sum_{j=0}^{k} \int_s^x \int_t^y a(\xi,\eta)K_j(x,y,\xi,\eta)d\eta d\xi \le \int_s^x \int_t^y a(\xi,\eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta)d\beta d\alpha\right) d\eta d\xi \tag{1.2.5}$$

which implies

$$\sum_{j=0}^{k+1} K_j(x,y,s,t) \leq 1 + \int_s^x \int_t^y a(\xi,\eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right) d\eta d\xi \tag{1.2.6}$$

In order to complete the inductive argument (and also the proof) it is sufficient to prove that

$$1 + \int_s^x \int_t^y a(\xi,\eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right) d\eta d\xi \leq \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\eta d\xi \right).$$

Consider the function

$$G(\xi,\eta) = \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right). \tag{1.2.7}$$

For this function we have

$$\frac{\partial G}{\partial \xi}(\xi,\eta) = -\int_\eta^y a(x,\beta) d\beta \cdot G(x,y) \text{ and}$$

$$\frac{\partial^2 G}{\partial \xi \partial \eta}(\xi,\eta) = a(\xi,\eta) \cdot G(x,y) + \int_\xi^x a(\alpha,\eta) d\alpha \cdot \int_\eta^y a(\xi,\beta) d\beta \cdot G(\xi,\eta).$$

From this equality and the nonnegativity of a we obtain

$$a(\xi,\eta) G(\xi,\eta) \leq \frac{\partial^2 G}{\partial x \partial y}(\xi,\eta),$$

hence

$$\int_s^x \int_t^y a(\xi,\eta) G(\xi,\eta)) d\eta d\xi \leq \int_s^x \int_t^y \frac{\partial^2 G}{\partial \xi \partial \eta}(\xi,\eta) d\eta d\xi.$$

But calculating the integrals from the right hand side expression we obtain

$$-1 + \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\eta d\xi \right),$$

so the proof is complete. □

1.3 The abstract comparison lemma

The proof of theorem 2.1 in [1] contains an error on line 7. The quantity $n_y(x,y)+w_y(x,y)$, where $n(x,y) = \int\limits_0^x \int\limits_0^y a(s,t)u(s,t)dtds$ is not necessary nonnegative, hence the inequality is not valid.

In the following we use our new tool, the abstract Gronwall-comparison lemma, Lemma 0.3.2 to prove the theorem 2.1 from [1].

Theorem 1.3.1 ([1],[17]). *Suppose $u(x,y)$, $w(x,y)$ and $a(x,y)$ are non-negative continuous functions defined on a domain D. If inequality* (1.1.1) *is satisfied for all $(x,y) \in D$, then*

$$u(x,y) \leq w(x,y) + G^{-1}\left(\int_0^x \int_0^y a(s,t)dtds\right), \ \forall (x,y) \in D, \tag{1.3.1}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{s+w}, r \geq r_0 > 0, \tag{1.3.2}$$

G^{-1} is the inverse function of G and $\int_0^x \int_0^y a(s,t)dtds \in Dom(G^{-1}),\ \forall (x,y) \in D$.

Proof. We can see in [17], that if we take the integral operator $A_1 : C(D) \to C(D)$ defined by

$$A_1(u)(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)u(s,t)dtds \tag{1.3.3}$$

it's a PO, but its fixed point is not the solution of the integral equation

$$u(x,y) = w(x,y) + G^{-1}\left(\int_0^x \int_0^y a(s,t)dtds\right), \ \forall (x,y) \in D, \tag{1.3.4}$$

Now we have to find a PO $A_2 : C(D) \to C(D)$, with the property $A_1 \leq A_2$ and with the fixed point, which satisfies the equation (1.3.4) and due to Lemma 0.3.2 we will finish the proof of the theorem.

From the equation (1.3.4) we get:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial w}{\partial x}(x,y) + (G^{-1})'\left(\int_0^x \int_0^y a(s,t)dtds\right)\int_0^y a(x,t)dt, \forall (x,y) \in D.$$

But

$$(G^{-1})'\left(\int_0^x \int_0^y a(s,t)dtds\right) = u(x,y), \forall (x,y) \in D,$$

so we have

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial w}{\partial x}(x,y) + u(x,y)\int_0^y a(x,t)dt, \forall (x,y) \in D.$$

From the basic theorem of the calculus we have:

$$u(x,y) - u(0,y) = \int_0^x \frac{\partial u}{\partial x}(s,y)ds = \int_0^x \left(\frac{\partial w}{\partial x}(s,y) + u(s,y)\int_0^y a(s,t)dt\right)ds.$$

From here we get the PO $A_2 : C(D) \to C(D)$, defined by

$$A_2(u)(x,y) = w(x,y) + \int_0^y \int_0^y u(s,y)a(s,t)dtds + G^{-1}(0),$$

with the fixed point satisfying the equation (1.3.4), and obviously $A_1 \leq A_2$, so with the Lemma 0.3.2 the proof is complete. $\square$

Now we present some nonlinear Bihari-Wendroff type inequality from [8].

Theorem 1.3.2 (Sz. András and A. Mészáros - [8]). *Let $u(x,y)$, $w(x,y)$ and $a(x,y)$ non-negative continuous functions defined on a domain D, and let $g:[0,\infty)\to(0,\infty)$ a continuous non-decreasing function, and suppose that $w(x,y)\le u(x,y), \forall (x,y)\in D$. If*

$$u(x,y)\le w(x,y)+\int_0^x\int_0^y a(s,t)g(u(s,t))dtds, \forall (x,y)\in D \tag{1.3.5}$$

then

$$u(x,y)\le G^{-1}\left(G(w(x,y))+\int_0^x\int_0^y a(s,t)dtds\right), \forall (x,y)\in D, \tag{1.3.6}$$

where

$$G(r)=\int_{r_0}^r \frac{ds}{g(s)}, r\ge r_0>0 \tag{1.3.7}$$

G^{-1} is the inverse function of G and $G(w(x,y))+\int_0^x\int_0^y a(s,t)dtds\in Dom(G^{-1}), \forall (x,y)\in D$.

Proof. Let us consider the integral operator $A_1: C(D)\to C(D)$ defined by the right side of the the inequality (1.3.5), namely

$$A_1(u)(x,y)=w(x,y)+\int_0^x\int_0^y a(s,t)g(u(s,t))dtds, \forall (x,y)\in D,$$

and the function

$$u(x,y)=G^{-1}\left(G(w(x,y))+\int_0^x\int_0^y a(s,t)dtds\right), \forall (x,y)\in D. \tag{1.3.8}$$

We can easily check that A_1 is a PO, but its fixed point doesn't satisfy the equation (1.3.8), so we cannot use the first abstract Gronwall lemma, we have to deal with the abstract Gronwall-comparison lemma, like in the proof of previous theorem.

From the (1.3.8) we have:

$$\frac{\partial u}{\partial x}(x,y)=(G^{-1})'\left(G(w(x,y))+\int_0^x\int_0^y a(s,t)dtds\right)\left(G'(w(x,y))\frac{\partial w}{\partial x}(x,y)+\int_0^y a(x,t)dt\right)$$

But

$$(G^{-1})'\left(G(w(x,y))+\int_0^x\int_0^y a(s,t)dtds\right)=g(u(x,y)),$$

so

$$\frac{\partial u}{\partial x}(x,y)=g(u(x,y))\left(\frac{1}{g(w(x,y))}\frac{\partial w}{\partial x}(x,y)+\int_0^y a(x,t)dt\right).$$

From the basic theorem of calculus we have:

$$u(x,y)-u(0,y)=\int_0^x\frac{\partial u}{\partial x}(s,y)ds=\int_0^x\frac{g(u(s,y))}{g(w(s,y))}\frac{\partial w}{\partial x}(s,y)ds+\int_0^x\int_0^y a(s,t)g(u(s,y))dtds$$

This relation shows that the function u defined by (1.3.8) is the fixed point of the integral operator $A_2: C(D)\to C(D)$, defined by

$$A_2(u)(x,y) = w(0,y)+\int_0^x \frac{g(u(s,y))}{g(w(s,y))}\frac{\partial w}{\partial x}(s,y)ds+\int_0^x\int_0^y a(s,t)g(u(s,y))dtds, \forall (x,y)\in D.$$

If we want to apply the abstract Gronwall-comparison lemma, we need to consider the set

$$X = \{u \in C(D)|\text{u increasing in y and } \ u(0,y) = w(0,y), u(x,0) = w(x,0)\}$$

and the restrictions of A_1, A_2 to X. This is necessary in order to obtain $A_1u \leq A_2u$. But we do not know $A_2u \in X$, and hence we can not apply the abstract Gronwall comparison lemma as stated in [34] or [41]. This difficulty can be overcame if we observe that:

- from $u \leq A_1u$ we deduce $u \leq u^*$, where u^* is the limit of successive approximation sequence for the operator A_1 starting from u;
- if $\bar{u}$ is the fixed point of A_2, then it is sufficient to have $A_1(\bar{u}) \leq A_2(\bar{u})$.

Indeed if A_1 is a PO, then u^* is the limit of the successive approximation sequence starting from $\bar{u}$ and from $A_1(\bar{u}) \leq A_2(\bar{u})$ we can prove by induction that $A_1^k(\bar{u}) \leq \bar{u}$, so $u^* \leq \bar{u}$. Due to this observation it is sufficient to prove $A_1\bar{u} \leq A_2\bar{u}$. But $\bar{u}$ is defined by (1.3.8), so $w(x,y) \leq \bar{u}(x,y)$, hence $g(w(s,y)) \leq g(\bar{u}(s,y))$, $\forall 0 \leq s \leq x$, so

$$w(x,y) \leq w(0,y) + \int_0^x \frac{g(\bar{u}(s,y))}{g(w(s,y))}\frac{\partial w}{\partial x}(s,y)ds. \tag{1.3.9}$$

From (1.3.8) we can deduce that $\bar{u}$ is nondecreasing in the second variable, hence

$$\int_0^x\int_0^y a(s,t)g(\bar{u}(s,t))dtds \leq \int_0^x\int_0^y a(s,t)g(\bar{u}(s,y))dtds, \forall (x,y)\in D. \tag{1.3.10}$$

From (1.3.9) and (1.3.10) we deduce $A_1(\bar{u}) \leq A_2(\bar{u})$, so the proof is complete.

□

Chapter 2

Wendroff type inequalities on time scales

In this chapter actually we extend our results form the previous chapter to arbitrary time scales. Recently some authors obtained results regarding to Wentroff type inequalities on arbitrary time scales (see [19], [4], [5]).

The results from [9] improve the known Wendroff type inequalities on time scales and here are also presented different proofs for the existing inequalities.

In what follows, we will present some results from [9], but at first let us recall some basic definitions and results about the time scale analysis.

2.1 Time scale analysis

The time scale calculus was founded by Stefan Hilger in his PhD thesis (see [20]) as a unification of the classical real analysis, the q-calculus and the theory of difference equations. Since then this theory has been extensively studied in order to obtain a better understanding and a unified viewpoint of mathematical phenomenons occurring in the theory of difference equations and in the theory of differential equations. For an excellent introduction to the calculus on time scales and to the theory of dynamic equations on time scales we recommend the books [13] and [14] by M. Bohner and A. Peterson. Throughout in this paper we use the basic notations from these books. For the sake of coherency we recall a few basic definitions, notations and theorems from [13].

Definition 2.1.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 2.1.2. We define the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ by the relations

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

Using these operators we can classify the points of time scale $\mathbb{T}$ as left dense, left scattered, right dense and right scattered according to whether $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ and $\sigma(t) > t$ respectively.

Definition 2.1.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point in $\mathbb{T}$. The set of all rd-continuous functions is denoted by C_{rd}. If $\mathbb{T}$ has left scattered maximum m, then

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus \{m\} & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases} \tag{2.1.1}$$

We define the graininess function $\mu : \mathbb{T}^{\kappa} \to \mathbb{R}$ by the relation

$$\mu(t) = \sigma(t) - t.$$

We also define for f the function $f^{\sigma} : \mathbb{T}^{\kappa} \to \mathbb{R}$ by

$$f^{\sigma}(t) = f(\sigma(t)), \ \forall t \in \mathbb{T}.$$

Definition 2.1.4. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number (provided if exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e. $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)\,[\sigma(t) - s]\right| \leq |\sigma(t) - s|, \ \forall s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

Theorem 2.1.5. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists and is a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Definition 2.1.6. A function $F : \mathbb{T} \to \mathbb{R}$ is said to be an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. We define the integral of f by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \tag{2.1.2}$$

where $s, t \in \mathbb{T}$.

Definition 2.1.7. The function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^\kappa$. We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ the set of all regressive and rd-continuous functions and define

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}. \tag{2.1.3}$$

Definition 2.1.8. For $p \in \mathcal{R}$ we define (see [13]) the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ as the unique solution to the scalar initial value problem

$$x^\Delta(t) = p(t)x(t), \quad x(t_0) = 1. \tag{2.1.4}$$

If $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$, for all $t \in \mathbb{T}$. We note that, if $\mathbb{T} = \mathbb{R}$, the exponential function is given by

$$e_p(t, s) = \exp\Big(\int_s^t p(\tau)d\tau\Big), \quad e_\alpha(t, s) = \exp(\alpha(t - s)), \quad e_\alpha(t, 0) = \exp(\alpha t), \tag{2.1.5}$$

for $s, t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function. To compare with the discrete case, if $\mathbb{T} = \mathbb{Z}$ (the set of integers), the exponential function is given by

$$e_p(t, s) = \prod_{\tau=s}^{t-1} [1 + p(\tau)], \quad e_\alpha(t, s) = (1 + \alpha)^{t-s}, \quad e_\alpha(t, 0) = (1 + \alpha)^t, \tag{2.1.6}$$

for $s, t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$.

Theorem 2.1.9 (Properties of the exponential function)**.** *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$;
(vi) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;
(vii) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$,
where for all $p, q \in \mathcal{R}$ we define

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t),$$

and

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)},$$

for all $t \in \mathbb{T}^\kappa$.

We remark, that $(\mathcal{R}, \oplus)$ is an Abelian group, called the regressive group.

2.2 Linear Wendroff type inequalities on time scales

In what follows we assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales with at least two points and we consider the time scale intervals $\tilde{\mathbb{T}}_1 = [a_1, \infty) \cap \mathbb{T}_1$ and $\tilde{\mathbb{T}}_2 = [a_2, \infty) \cap \mathbb{T}_2$, for $a_1 \in \mathbb{T}_1$ and $a_2 \in \mathbb{T}_2$. Let us denote $D = \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$. We also use the notation $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $e_p(t,s)$ denotes the usual exponential function on time scales with $p \in \mathcal{R}$, where p is a regressive function (see [13]). In [19] the authors obtained the following results:

Theorem 2.2.1. *(Theorem 2.1. in [19]) Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D, \mathbb{R}_0^+)$ with $w(t_1,t_2)$ nondecreasing in each of its variables. If*

$$u(t_1,t_2) \le w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{2.2.1}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \le w(t_1,t_2) e_{\int_{a_2}^{t_2} a(t_1,s_2)\Delta_2 s_2}(t_1,a_1), \quad (t_1,t_2) \in D. \tag{2.2.2}$$

Theorem 2.2.2. *(Theorem 2.2. in [19]) Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D, \mathbb{R}_0^+)$, with $w(t_1,t_2)$ and $a(t_1,t_2)$ nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S, \mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in D \times D : a_1 \le s_1 \le t_1, a_2 \le s_2 \le t_2\}$ and g is nondecreasing in the first two variables. If u satisfies the condition*

$$u(t_1,t_2) \le w(t_1,t_2) + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{2.2.3}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \le w(t_1,t_2) e_{\int_{a_2}^{t_2} a(t_1,t_2)g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1), \ \forall (t_1,t_2) \in D. \tag{2.2.4}$$

In what follows we present the improvements of (2.2.2) and (2.2.4) obtained in [9]. These improved versions imply also improved bounds in the nonlinear cases (see Theorem 3.1 and 3.2 in [19]). Applying the same technique we can obtain new (and simple) proofs for the previous theorems too.

But at first we need some mathematical tools to define a good notion of metric space of the continuous functions defined on times scale pairs, similar to the notion of the usual metric space of continuous functions equipped with some metric of Bielecki type.

2.2.1 Some mathematical tools

In this section we recall some preliminary results from [9] regarding to the metrics of Bielecki type and differentiation under the integral sign.

We extend the metric introduced by C.C. Tisdell and A. Zaidi in [42] to functions with several variables. This allows us to prove that our operators are Picard operators, in fact they are contractions if we use a well chosen metric. Suppose that $\alpha, \beta > 0$ are real constants and define the functionals

$$d_{\alpha,\beta} : C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n) \times C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)) \to \mathbb{R} \tag{2.2.5}$$

by

$$d_{\alpha,\beta}(u, v) = \sup_{\substack{s_1 \in [a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \\ s_2 \in [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}}} \frac{\|u(s_1, s_2) - v(s_1, s_2)\|}{e_\alpha(s_1, a_1) \cdot e_\beta(s_2, a_2)} \tag{2.2.6}$$

for all $u, v \in C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ and

$$\| \cdot \|_{\alpha,\beta} : C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n) \to \mathbb{R} \tag{2.2.7}$$

$$\|u\|_{\alpha,\beta} = \sup_{\substack{s_1 \in [a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \\ s_2 \in [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}}} \frac{\|u(s_1, s_2)\|}{e_\alpha(s_1, a_1) \cdot e_\beta(s_2, a_2)} \tag{2.2.8}$$

for all $u \in C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$, where $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$.

Lemma 2.2.3 ([9]). *If $\alpha, \beta > 0$, and $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$, we have the following properties:*

1. *$d_{\alpha,\beta}$ is a metric on $C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$;*
2. *$C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ is a complete metric space with $d_{\alpha,\beta}$;*
3. *$\| \cdot \|_{\alpha,\beta}$ is a norm on $C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ and it is equivalent to $\| \cdot \|_{0,0}$.*
4. *$(C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n), \| \cdot \|_{\alpha,\beta})$ is a Banach space.*

The proof of this lemma is quite straightforward, so we omit it. For the simplicity of notation in what follows we denote $C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R})$ by X. Using this Bielecki type (or "TZ") metric, we prove the following properties:

Theorem 2.2.4 ([9]). *If $w, a \in X$, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$, the operator $A_1 : X \to X$ defined by*

$$A_1(u)(t_1, t_2) = w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{2.2.9}$$

is well defined and there exist $\alpha, \beta > 0$ such that A_1 is a contraction on $(X, d_{\alpha,\beta})$.

Theorem 2.2.5 ([9]). *If $w, a \in X$, g is continuous, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$, the operator $A_2 : X \to X$ defined by*

$$A_2(u)(t_1, t_2) = w(t_1, t_2) + a(t_1, t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1, t_2, s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{2.2.10}$$

is well defined and there exist $\alpha, \beta > 0$ such that A_2 is a contraction on $(X, d_{\alpha,\beta})$.

Proof of theorem 2.2.4. Denote by M the maximum of $a(s_1, s_2)$ if $s_1 \in [a_1, \sigma_1(b_1)]_{\mathbb{T}_1}$ and $s_2 \in [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}$. Due to the given conditions M exists and $M < \infty$.

$$|A_1(u)(t_1, t_2) - A_1(v)(t_1, t_2)| \leq \int_{a_1}^{t_1} \int_{a_2}^{t_2} |a(s_1, s_2)||u(s_1, s_2) - v(s_1, s_2)|\Delta_1 s_1 \Delta_2 s_2$$

$$\begin{aligned}
&\leq M \int_{a_1}^{t_1} \int_{a_2}^{t_2} \frac{|u(s_1, s_2) - v(s_1, s_2)|}{e_\alpha(s_1, a_1) e_\beta(s_2, a_2)} e_\alpha(s_1, a_1) e_\beta(s_2, a_2) \Delta_1 s_1 \Delta_2 s_2 \\
&\leq M \|u - v\|_{\alpha,\beta} \int_{a_1}^{t_1} \int_{a_2}^{t_2} e_\alpha(s_1, a_1) e_\beta(s_2, a_2) \Delta_1 s_1 \Delta_2 s_2 \\
&\leq \frac{M}{\alpha\beta} \|u - v\|_{\alpha,\beta} e_\alpha(t_1, a_1) e_\beta(t_2, a_2).
\end{aligned}$$

The last inequality implies

$$\|A_1(u) - A_1(v)\|_{\alpha,\beta} \leq \frac{M}{\alpha\beta} \|u - v\|_{\alpha,\beta}, \tag{2.2.11}$$

so A_1 is a contraction on X if $\alpha\beta > M$. □

Remark 2.2.6. The proof of theorem 2.2.5 can be done in a similar way by using the maximum of g on $([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2})^2$.

Remark 2.2.7. We can obtain the contractive property of a more general nonlinear operator $A_3 : X \to X$ defined by

$$A_3(u)(t_1, t_2) = w(t_1, t_2) + a(t_1, t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(t_1, t_2, s_1, s_2, u(s_1, s_2)) \Delta_1 s_1 \Delta_2 s_2,$$

where f is continuous and has the Lipschitz property in the last variable.

Remark 2.2.8. Due to theorem 2.2.4 and 2.2.5 the operators A_1 and A_2 are Picard operators.

In the calculations we use the following two properties:

Lemma 2.2.9 ([9])**.** *If f is continuous and is continuously Δ differentiable with respect to t, then the function*

$$U(t) = \int_a^t f(s, t) \Delta s$$

admits a Δ derivative with respect to t and

$$U^\Delta(t) = \int_a^t \frac{\partial f}{\Delta t}(s, t) \Delta s + f(t, t).$$

Lemma 2.2.10 ([9])**.** *If $f : E \to \mathbb{R}$ is a continuous function, where*

$$E = \{(s, t) \in \mathbb{T}_1 \times \mathbb{T}_2 | a \leq t < b, a \leq s < t\},$$

then the function $g : [a, b) \to \mathbb{R}$, *defined by*

$$g(t) = \int_a^t f(s, t)\Delta_1 s$$

is Δ *integrable on* $[a, b)$ *and we have*

$$\int_a^b \int_a^t f(s, t)\Delta_1 s \Delta_2 t = \int_a^b \int_{\sigma(s)}^b f(s, t)\Delta_2 t \Delta_1 s.$$

Remark 2.2.11. If $f : E \to \mathbb{R}$ is a continuous function, where

$$E = \{(t_1, t_2, s_1, s_2) \in (\mathbb{T}_1 \times \mathbb{T}_2)^2 | a_1 \le s_1 < t_1, a_2 \le s_2 < t_2\},$$

then the function $g : [a_1, t_1) \times [a_2, t_2) \to \mathbb{R}$, defined by

$$g(s_1, s_2) = \int_{a_1}^{s_1} \int_{a_2}^{s_2} f(t_1, t_2, \xi_1, \xi_2)\Delta_1 \xi_1 \Delta_2 \xi_2$$

is Δ integrable on $[a_1, t_1) \times [a_2, t_2)$ and we have

$$\int_{a_1}^{t_1} \int_{a_2}^{t_2} g(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2 =$$

$$= \int_{a_1}^{t_1} \int_{a_2}^{t_2} \int_{\sigma_1(s_1)}^{t_1} \int_{\sigma_2(s_2)}^{t_2} f(\xi_1, \xi_2, s_1, s_2)\Delta_1 \xi_1 \Delta_2 \xi_2 \Delta_1 s_1 \Delta_2 s_2.$$

2.2.2 The improvements of the linear inequalities

In this section we give new estimates for u and we prove that these are better than (2.2.2), (2.2.4). These results are from [9]. We need the following lemma

Lemma 2.2.12 ([9])**.** *For the function* $V : E \to \mathbb{R}$, *defined by*

$$V(t_1, t_2, s_1, s_2) = e_{\int_{s_2}^{t_2} a(t_1, \xi_2)\Delta_2 \xi_2}(t_1, s_1),$$

where

$$E = \{(t_1, t_2, s_1, s_2) \in (\mathbb{T}_1 \times \mathbb{T}_2)^2 | a_1 \le s_1 < t_1, a_2 \le s_2 < t_2\}$$

we have

$$a(s_1, s_2)V(t_1, t_2, \sigma_1(s_1), \sigma_2(s_2)) \le \frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1, t_2, s_1, s_2). \tag{2.2.12}$$

and

$$a(t_1, t_2)V(t_1, t_2, s_1, s_2) \le \frac{\partial^2 V}{\Delta_1 t_1 \Delta_2 t_2}(t_1, t_2, s_1, s_2). \tag{2.2.13}$$

Proof. The function V is Δ_1 differentiable with respect to s_1 and we have

$$\frac{\partial V}{\Delta_1 s_1}(t_1,t_2,s_1,s_2) = -\int\limits_{s_2}^{t_2} a(s_1,\xi_2)\Delta_2\xi_2 \cdot V(t_1,t_2,\sigma_1(s_1),s_2).$$

Moreover the function $\frac{\partial V}{\Delta_1 s_1}$ is Δ_2 differentiable and we have

$$\frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2) = \int\limits_{s_2}^{t_2} a(s_1,\xi_2)\Delta_2\xi_2 \int\limits_{\sigma_1(s_1)}^{t_1} a(\xi_1,s_2)\Delta_1\xi_1 \cdot V(t_1,t_2,\sigma_1(s_1),\sigma(s_2)) +$$
$$+a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)).$$

Since the function a is nonnegative we obtain

$$a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)) \leq \frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2).$$

Using a similar argument we have

$$\frac{\partial V}{\Delta_1 t_1}(t_1,t_2,s_1,s_2) = \int\limits_{s_2}^{t_2} a(t_1,\xi_2)\Delta_2\xi_2 \cdot V(t_1,t_2,s_1,s_2).$$

The function $\frac{\partial V}{\Delta_1 t_1}$ is Δ_2 differentiable with respect to t_2 and we have

$$\frac{\partial^2 V}{\Delta_1 t_1 \Delta_2 t_2}(t_1,t_2,s_1,s_2) = \int\limits_{s_2}^{\sigma_2(t_2)} a(t_1,\xi_2)\Delta_2\xi_2 \int\limits_{s_1}^{t_1} a(\xi_1,t_2)\Delta_1\xi_1 \cdot V(t_1,t_2,s_1,s_2) +$$
$$+a(t_1,t_2)V(t_1,t_2,s_1,s_2).$$

Since the function a is nonnegative we obtain

$$a(t_1,t_2)V(t_1,t_2,s_1,s_2) \leq \frac{\partial^2 V}{\Delta_1 t_1 \Delta_2 t_2}(t_1,t_2,s_1,s_2).$$

□

Theorem 2.2.13 (Sz. András and A. Mészáros - [9])**.** *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$ with $w(t_1,t_2)$ nondecreasing in each of its variables. If $u(t_1,t_2)$ satisfies*

$$u(t_1,t_2) \leq w(t_1,t_2) + \int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{2.2.14}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \leq w(t_1,t_2) + \int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1 \Delta_2 s_2, \tag{2.2.15}$$

for $(t_1,t_2) \in D$, where σ_1 and σ_2 are the jump operators on $\mathbb{T}_1$ respectively $\mathbb{T}_2$.

Proof. The integral operator $A : C(D) \to C(D)$ defined by

$$A(u)(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2, \tag{2.2.16}$$

is a Picard operator (due to theorem 2.2.4). Moreover the space $(C(D), \|\cdot\|)$ is an ordered Banach space with the natural ordering

$$u \le v \Leftrightarrow u(t_1,t_2) \le v(t_1,t_2),\ \forall (t_1,t_2) \in D$$

and the operator A is an increasing operator, so the inequality $u \le Au$ implies $u \le u^*$, where u^* is the unique solution of the equation $Au = u$. On the other hand it is easy to check that the unique fixed point of A is not the function

$$\overline{u}(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1\Delta_2 s_2,$$

so by Lemma 0.3.3 we need to prove $A\overline{u} \le \overline{u}$. Using the function V from Lemma 2.2.12 it is sufficient to prove

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)\Delta_1 s_1\Delta_2 s_2 + \tag{2.2.17}$$

$$+\int_{a_1}^{t_1}\int_{a_2}^{t_2}\int_{a_1}^{s_1}\int_{a_2}^{s_2} a(s_1,s_2)a(\xi_1,\xi_2)w(\xi_1,\xi_2)V(s_1,s_2,\sigma_1(\xi_1),\sigma_2(\xi_2))\Delta_1\xi_1\Delta_2\xi_2\Delta_1 s_1\Delta_2 s_2 \le \tag{2.2.18}$$

$$\le \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2. \tag{2.2.19}$$

Changing the order of integration in (2.2.18) and renaming the variables it is sufficient to prove

$$1 + \int_{\sigma_1(s_1)}^{t_1}\int_{\sigma_2(s_2)}^{t_2} a(\xi_1,\xi_2)V(\xi_1,\xi_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1\xi_1\Delta_2\xi_2 \le V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)).$$

This can be obtained by integrating (2.2.13) from $\sigma_1(s_1)$ to t_1 and than from $\sigma_2(s_2)$ to t_2. $\square$

Theorem 2.2.14 (Sz. András and A. Mészáros - [9])**.** *If the conditions of Theorem 2.2.13 are satisfied, the estimation of the Theorem 2.2.13 is better than the estimation from Theorem 2.2.1.*

Proof. Integrating inequality (2.2.12) with respect to s_1 and s_2 on the rectangle $[a_1,t_1)_{\mathbb{T}_1} \times [a_2,t_2)_{\mathbb{T}_2}$ we deduce

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \le$$

$$\le \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 =$$

$$= V(t_1,t_2,t_1,t_2) - V(t_1,t_2,a_1,t_2) - V(t_1,t_2,t_1,a_2) + V(t_1,t_2,a_1,a_2).$$

But $V(t_1,t_2,t_1,t_2) = V(t_1,t_2,a_1,t_2) = V(t_1,t_2,t_1,a_2) = 1$, so we obtain

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1 \Delta_2 s_2 \le V(t_1,t_2,a_1,a_2) - 1. \tag{2.2.20}$$

The function w is nonnegative and nondecreasing in both variables, hence we have

$$w(t_1,t_2) + \int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1 \Delta_2 s_2 \le$$

$$w(t_1,t_2)\left(1 + \int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} a(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1 \Delta_2 s_2\right) \le$$

$$\le w(t_1,t_2)V(t_1,t_2,a_1,a_2).$$

This inequality shows that the estimation in Theorem 2.2.13 is better than the estimation from Theorem 2.2.1. □

Lemma 2.2.15 ([9]). *For the function $W : E \to \mathbb{R}$, defined by*

$$W(t_1,t_2,s_1,s_2) = e_{\int_{s_2}^{t_2} a(t_1,t_2)g(t_1,t_2,t_1,\eta)\Delta_2\eta}(t_1,s_1),$$

where

$$E = \{(t_1,t_2,s_1,s_2) \in (\mathbb{T}_1 \times \mathbb{T}_2)^2 | a_1 \le s_1 < t_1, a_2 \le s_2 < t_2\}$$

we have

$$a(t_1,t_2)g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)) \le \frac{\partial^2 W}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2) \tag{2.2.21}$$

Remark 2.2.16. The proof of the Lemma 2.2.15 is similar to the proof of the Lemma 2.2.12.

Theorem 2.2.17 (Sz. András and A. Mészáros - [9]). *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$, with $w(t_1,t_2)$ and $a(t_1,t_2)$ nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S,\mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in D \times D : a_1 \le s_1 \le t_1, a_2 \le s_2 \le t_2\}$ and g is nondecreasing in the first two variables. If u satisfies the condition*

$$u(t_1,t_2) \le w(t_1,t_2) + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{2.2.22}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \leq w(t_1,t_2) + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)w(s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2, \quad (2.2.23)$$

for $(t_1,t_2)\in D$, *where* σ_1 *and* σ_2 *are the jump operators on* $\mathbb{T}_1$ *respectively* $\mathbb{T}_2$.

Proof. We apply the same technique as in [19]. We consider that t_1^* and t_2^* are fixed and we consider the operator $A^* : C(D) \to C(D)$ defined by

$$A^*(u)(t_1,t_2) = w(t_1,t_2) + a(t_1^*,t_2^*)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1^*,t_2^*,s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2. \quad (2.2.24)$$

It is clear that if u satisfies the conditions of theorem 2.2.17, then $u(t_1,t_2)\leq A^*(u)(t_1,t_2)$, for $t\leq t_1^*$ and $t_2\leq t_2^*$. t_1^*, t_2^* beeing fixed, theorem 2.2.13 implies

$$u(t_1,t_2) \leq w(t_1,t_2)+ \quad (2.2.25)$$

$$+\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(t_1^*,t_2^*)g(t_1^*,t_2^*,s_1,s_2)w(s_1,s_2)H(s_1,s_2,t_1,t_2,t_1^*,t_2^*)\Delta_1 s_1\Delta_2 s_2,$$

where

$$H(s_1,s_2,t_1,t_2,t_1^*,t_2^*) = e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))$$

This inequality is valid for $t_1 = t_1^*$ and $t_2 = t_2^*$ and t_1^*, t_2^* are arbitrary, so we obtain (2.2.23). □

Theorem 2.2.18 (Sz. András and A. Mészáros - [9])**.** *If the conditions of Theorem 2.2.17 are satisfied, the estimation of the Theorem 2.2.17 is better than the estimation from Theorem 2.2.2.*

Proof. Integrating inequality (2.2.21) from a_1 to t_1 and from a_2 to t_2 with respect to s_1 and s_2 on the rectangle $[a_1,t_1)_{\mathbb{T}_1}\times[a_2,t_2)_{\mathbb{T}_2}$ we have

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(t_1,t_2)g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \leq$$

$$\leq \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{\partial^2 W}{\Delta_1 s_1\Delta_2 s_2}(t_1,t_2,s_1,s_2)\Delta_1 s_1\Delta_2 s_2 =$$

$$= W(t_1,t_2,t_1,t_2) - W(t_1,t_2,a_1,t_2) - W(t_1,t_2,t_1,a_2) + W(t_1,t_2,a_1,a_2).$$

But $W(t_1,t_2,t_1,t_2) = W(t_1,t_2,a_1,t_2) = W(t_1,t_2,t_1,a_2) = 1$, so we obtain

$$a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \leq \quad (2.2.26)$$

$$\leq W(t_1, t_2, a_1, a_2) - 1.$$

The function w is nonnegative and nondecreasing in both variables, hence we have

$$w(t_1, t_2) + a(t_1, t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1, t_2, s_1, s_2) w(s_1, s_2) W(t_1, t_2, \sigma_1(s_1), \sigma_2(s_2)) \Delta_1 s_1 \Delta_2 s_2 \leq$$

$$w(t_1, t_2) \left(1 + a(t_1, t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1, t_2, s_1, s_2) W(t_1, t_2, \sigma_1(s_1), \sigma_2(s_2)) \Delta_1 s_1 \Delta_2 s_2 \right) \leq$$

$$\leq w(t_1, t_2) W(t_1, t_2, a_1, a_2).$$

This inequality shows that the estimation in Theorem 2.2.17 is better than the estimation from Theorem 2.2.2. $\square$

2.3 Nonlinear inequalities

In this section we present the improved estimations (see [9])to the recently proved nonlinear integral inequalities in ([19]) combining the method from ([19]) with theorem 2.2.13 and 2.2.17. First we recall the nonlinear integral inequalities from [19]:

Theorem 2.3.1. *(Theorem 3.1 in [19]) Let $u(t_1, t_2)$, $w(t_1, t_2)$, $a(t_1, t_2) \in C(D, \mathbb{R}_0^+)$ with $w(t_1, t_2)$ nondecreasing in each of its variables. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1, t_2) \leq w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) u^q(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2 \tag{2.3.1}$$

for $(t_1, t_2) \in D$, then

$$u(t_1, t_2) \leq w^{\frac{1}{p}}(t_1, t_2) \left[e_{\int_{a_2}^{t_2} a(t_1, s_2) w^{\frac{q}{p}-1}(t_1, s_2) \Delta_2 s_2}(t_1, a_1) \right]^{\frac{1}{p}}, (t_1, t_2) \in D. \tag{2.3.2}$$

Theorem 2.3.2. *(Theorem 3.2 in [19]) Let $u(t_1, t_2)$, $w(t_1, t_2)$, $a(t_1, t_2) \in C(D, \mathbb{R}_0^+)$, with $w(t_1, t_2)$ and $a(t_1, t_2)$ nondecreasing in each of the variables and $g(t_1, t_2, s_1, s_2) \in C(S, \mathbb{R}_0^+)$, where $S = \{(t_1, t_2, s_1, s_2) \in D \times D : a_1 \leq s_1 \leq t_1, a_2 \leq s_2 \leq t_2\}$ and g is nondecreasing in each of its variables. If p and q are two positive real numbers such that $p \geq q$ and is u satisfies the condition*

$$u^p(t_1, t_2) \leq w(t_1, t_2) + a(t_1, t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1, t_2, s_1, s_2) u^q(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{2.3.3}$$

for $(t_1, t_2) \in D$, then

$$u(t_1, t_2) \leq w^{\frac{1}{p}}(t_1, t_2) \left[e_{\int_{a_2}^{t_2} a(t_1, t_2) w^{\frac{q}{p}-1}(t_1, s_2) g(t_1, t_2, t_1, s_2) \Delta_2 s_2}(t_1, a_1) \right]^{\frac{1}{p}}, \ \forall (t_1, t_2) \in D. \tag{2.3.4}$$

In what follows we prove the following improvements of these to theorems:

Theorem 2.3.3 (Sz. András and A. Mészáros - [9])**.** *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$ with $w(t_1,t_2)$ nondecreasing in each of its variables. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1,t_2) \leq w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1\Delta_2 s_2 \tag{2.3.5}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \leq \left[w(t_1,t_2) + w(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} H(t_1,t_2,s_1,s_2)\Delta_1 s_1\Delta_2 s_2\right]^{\frac{1}{p}}, \tag{2.3.6}$$

where

$$H(t_1,t_2,s_1,s_2) = a(s_1,s_2)w^{\frac{q}{p}-1}(s_1,s_2)e_{\int_{\sigma(s_2)}^{t_2} a(t_1,\eta)w^{\frac{q}{p}-1}(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s1)),$$

$(t_1,t_2) \in D$.

Proof. Suppose $w(t_1,t_2) > 0$, $(t_1,t_2) \in D$. We denote u^p by $\overline{u}$. If u satisfies the conditions of the previous theorem, due to the monotonicity of w we obtain

$$\frac{\overline{u}(t_1,t_2)}{w(t_1,t_2)} \leq 1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{a(s_1,s_2)}{w(s_1,s_2)}\overline{u}^{\frac{q}{p}}(s_1,s_2)\Delta_1 s_1\Delta_2 s_2,$$

hence for the function v defined by the right hand side of the previous inequality we have

$$\frac{\partial^2 v}{\Delta_1 t_1\Delta_2 t_2} = \frac{a(t_1,t_2)}{w(t_1,t_2)}\overline{u}^{\frac{q}{p}}(t_1,t_2) \leq a(t_1,t_2)w^{\frac{q}{p}-1}(t_1,t_2)v(t_1,t_2).$$

Integrating both sides we deduce that the function v satisfies the following inequality:

$$v(t_1,t_2) \leq 1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w^{\frac{q}{p}-1}(s_1,s_2)v(s_1,s_2)\Delta_1 s_1\Delta_2 s_2.$$

Aplying theorem 2.2.13 for v and using $u(t_1,t_2) \leq (w(t_1,t_2)v(t_1,t_2))^{\frac{1}{p}}$ we obtain (2.3.6). $\square$

Remark 2.3.4. If $w \geq 0$, we can replace w with $w_\varepsilon = w + \varepsilon$ and then consider $\varepsilon \to 0$.

Remark 2.3.5. Due to Theorem 2.2.14 the estimation in Theorem 2.3.3 is better than the estimation in Theorem 2.3.1.

Using the same argument as in the previous theorem we obtain the following result:

Theorem 2.3.6 (Sz. András and A. Mészáros - [9]). *Let $u(t_1,t_2),\ w(t_1,t_2),\ a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$, with $w(t_1,t_2)$ and $a(t_1,t_2)$ nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S,\mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in D\times D : a_1 \le s_1 \le t_1, a_2 \le s_2 \le t_2\}$ and g is nondecreasing in the first two variables. If p and q are two positive real numbers such that $p \ge q$ and u satisfies the condition*

$$u^p(t_1,t_2) \le w(t_1,t_2) + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1\Delta_2 s_2, \tag{2.3.7}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \le w^{\frac{1}{p}}(t_1,t_2)\left[1 + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)H(t_1,t_2,s_1,s_2)\Delta_1 s_1\Delta_2 s_2\right]^{\frac{1}{p}}, \tag{2.3.8}$$

where

$$H(t_1,t_2,s_1,s_2) = w^{\frac{q}{p}-1}(s_1,s_2)e_{\int_{\sigma(s_2)}^{t_2} a(t_1,\eta)g(t_1,t_2,t_1,\eta)w^{\frac{q}{p}-1}(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s1)),$$

$(t_1,t_2) \in D$.

Remark 2.3.7. Due to Theorem 2.2.18 the estimation of the Theorem 2.3.6 is better than the estimation from Theorem 2.3.2.

Remark 2.3.8. Theorem 2.3.3 and Theorem 2.3.6 generalize and extend to time scales Theorem 2.1, Theorem 2.2 and Theorem 2.3. from [18].

2.4 Applications and examples

In what follows we present an application and two examples for Theorem 2.2.13 (see [9]). Let us consider the the following partial delta dynamic equation

$$\frac{\partial^2 u(t_1,t_2)}{\Delta_2 t_2\Delta_1 t_1} = F(t_1,t_2,u(t_1,t_2)) \tag{2.4.1}$$

on the domain D, equipped with the initial conditions

$$u(t_1,a_2) = g_1(t_1),\ u(a_1,t_2) = g_2(t_2), \forall t_1 \in \tilde{\mathbb{T}}_1,\ t_2 \in \tilde{\mathbb{T}}_2 \tag{2.4.2}$$

where $F \in C(D\times\mathbb{R}_0^+,\mathbb{R}_0^+)$, $g_1 \in C(\tilde{\mathbb{T}}_1,\mathbb{R}_0^+)$, $g_2 \in C(\tilde{\mathbb{T}}_2,\mathbb{R}_0^+)$ and g_1 and g_2 are nondecreasing.

If we assume, that F satisfies the inequality

$$F(t_1,t_2,u) \le f(t_1,t_2)u,\ \forall(t_1,t_2) \in D,\ u \in C(D,\mathbb{R}_0^+), \tag{2.4.3}$$

for a given function $f \in C(D,\mathbb{R}_0^+)$, which is nondecreasing in both of its variables.

Theorem 2.4.1 (Sz. András and A. Mészáros - [9]). *If u is the solution of the initial value problem* (2.4.1)-(2.4.2) *and the previous assumptions hold, then u satisfies the inequality*

$$\begin{aligned}u(t_1,t_2) \leq & g_1(t_1)+g_2(t_2)+ \\ & + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\left(g_1(s_1)+g_2(s_2)\right) e_{\int_{\sigma_2(s_2)}^{t_2} f(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1 \Delta_2 s_2. \quad (2.4.4)\end{aligned}$$

Proof. If $u(t_1,t_2)$ is a solution of the initial value problem (2.4.1)-(2.4.2), then it satisfies the equation

$$u(t_1,t_2) = g_1(t_1)+g_2(t_2)+\int_{a_1}^{t_1}\int_{a_2}^{t_2} F(s_1,s_2,u(s_1,s_2))\Delta_1 s_1\Delta_2 s_2.$$

From (2.4.3) we have

$$u(t_1,t_2) \leq g_1(t_1)+g_2(t_2)+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2.$$

Hence Theorem 2.2.13 can be applied for $w(t_1,t_2) := g_1(t_1)+g_2(t_2)$, $a(t_1,t_2) := f(t_1,t_2)$, $\forall (t_1,t_2)\in D$ and implies inequality (2.4.4). □

Remark 2.4.2. Using the previous theorem we can study also other properties of the solutions for the initial value problem (2.4.1)-(2.4.2) (existence, uniqueness, continuity, etc.). Moreover the same technique can be applied to a wide range of problems. If we replace (2.4.1)-(2.4.2) with

$$\frac{\partial^2 u^p(t_1,t_2)}{\Delta_2 t_2 \Delta_1 t_1} = F(t_1,t_2,u(t_1,t_2)) \quad (2.4.5)$$

on the domain D, and the initial conditions

$$u^p(t_1,a_2) = g_1(t_1),\ u^p(a_1,t_2) = g_2(t_2), \forall t_1 \in \tilde{\mathbb{T}}_1,\ t_2\in\tilde{\mathbb{T}}_2 \quad (2.4.6)$$

where we assume the same regularity conditions F, g_1, g_2, and also

$$F(t_1,t_2,u) \leq f(t_1,t_2)u^q,\ \forall (t_1,t_2)\in D,\ u\in C(D,\mathbb{R}_0^+), \quad (2.4.7)$$

for a fixed positive real numbers p and q with $p \geq q$, we could have a similar estimate for the solution u of the initial value problem (2.4.5)-(2.4.6), using our result form Theorem 2.3.3.

Example 2.4.3 ([9]). If $\mathbb{T}_1 := \mathbb{R}$, $\mathbb{T}_2 = \mathbb{R}$, $a \equiv 1$ and $w \equiv 1$, than by applying Theorem 2.2.13 we obtain

$$\begin{aligned}u(t_1,t_2) \leq & 1+\int_0^{t_1}\int_0^{t_2} \exp\left((t_1-s_1)(t_2-s_2)\right) ds_2 ds_1 \\ \leq & \sum_{k=1}^{\infty} \frac{(t_1 t_2)^{k+1}}{(k+1)\cdot(k+1)!}.\end{aligned}$$

On the other hand, if we use the estimate from Theorem 2.2.1, we have

$$u(t_1,t_2) \le \exp(t_1 t_2) = \sum_{k=0}^{\infty} \frac{(t_1 t_2)^k}{k!}.$$

It is clear that the difference between the two estimates increases exponentially and the ratio of the estimates tends to 0 as x tends to ∞. In order to obtain also a numerical comparison between the two estimates we calculated numerically the values of the difference $h_2(t_1,t_2) - h_1(t_1,t_2)$ on a grid with node points

$$0 = t_{1,0} < t_{1,1} < \cdots < t_{1,n-1} < t_{1,n} = 3.5,$$

$$0 = t_{2,0} < t_{2,1} < \cdots < t_{2,n-1} < t_{2,n} = 3.5,$$

where $\Delta t_1 = \Delta t_2 = 0.5$ and

$$h_1(t_1,t_2) := 1 + \int_0^{t_1}\int_0^{t_2} \exp\left((t_1-s_1)(t_2-s_2)\right) ds_2 ds_1,$$

$$h_2(t_1,t_2) := \exp(t_1 t_2).$$

$t_1 \setminus t_2$	0	0.5	1	1.5	2	2.5	3	3.5
0	0	0	0	0	0	0	0	0
0.5	0	0.01	0.07	0.19	0.40	0.70	1.16	1.80
1	0	0.07	0.40	1.16	2.70	5.60	10.82	20.02
1.5	0	0.19	1.16	3.93	10.82	26.90	63.16	143.4
2	0	0.40	2.70	10.82	35.93	109.41	318.81	906.65
2.5	0	0.70	5.60	26.90	109.41	414.74	1520.2	5476.4
3	0	1.16	10.82	63.16	318.81	1520.2	7067	32434
3.5	0	1.80	20.02	143.4	906.65	5476.4	32434	1.9021e+05

These numerical results show that the estimation from Theorem 2.2.13 are much sharper than the estimation from Theorem 2.2.1.

Example 2.4.4 ([9])**.** If $\mathbb{T}_1 := \mathbb{Z}$, $\mathbb{T}_2 = \mathbb{Z}$ and $a \equiv 1$, $w \equiv 1$, then for the solution of the inequality

$$u(m,n) \le 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} u(s,t)$$

we obtain (by applying Theorem 2.2.13) the estimation

$$u(m,n) \le 1 + m \cdot n^{m-1}.$$

For the same functions from Theorem 2.2.1 we deduce

$$u(m,n) \le (1+n)^m.$$

Due to the binomial theorem the first estimation is much sharper than the second one.

Chapter 3

Ulam-Hyers stability of dynamical equations on time scales

In this chapter we change a little bit the subject, but we will follow mainly the same rout, the operatorial approach. We will give another application of the Picard operators, i.e. the Ulam-Hyres stability of dynamical- and integral equations on time scales.

The definitions of Ulam-Hyers stability in the Introduction show that on an unbounded interval the Ulam-Hyers stability of the differential equation is not equivalent with the Ulam-Hyers stability of the corresponding integral equation (which is a fixed point problem), while on a bounded interval these two notions are the same.

So we remark that the usual operatorial technique with the Picard operators cannot be used in the usual way, when the domains of the dynamical equations are not bounded, in this case we will use some direct methods.

We will present some recent results from the paper [10].

Our results extend some recent results from [31],[32], [16], [29], [26] to dynamic equations and are more general than the results from [6].

3.1 Linear dynamic equations with constant coefficients

First we recall some examples regarding Ulam-Hyers stability of differential, difference and dynamic equations.

Example 3.1.1. ([16]) The differential equation $y' = ay, a \in \mathbb{R}$ is Ulam-Hyers stable if and only if $a \neq 0$. Moreover, the equation

$$y^{(n)} - \sum_{k=1}^{n} a_k y^{(n-k)} = 0,$$

where $a_j \in \mathbb{R}, 1 \leq j \leq n$ is Ulam-Hyers stable if and only if the roots of the corresponding characteristic equation are nonzero.

Example 3.1.2. ([32]) The difference equation $y_{n+1} = ay_n, a \in \mathbb{R}$ is Ulam-Hyers stable if and only if $|a| \neq 1$. Moreover, the equation

$$y_{n+p} = \sum_{k=0}^{p-1} a_k y_{n+k},$$

where $a_k \in \mathbb{R}, 0 \leq k \leq p-1$ is Ulam-Hyers stable if and only if the modulus of the roots of the corresponding characteristic equation are different from 1.

Example 3.1.3. If $\mathbb{T} = \{t_0, t_1, \ldots, t_n, \ldots\}$ with $t_0 = 1$, $\mu(t_{2j}) = \frac{1}{(2j+2)^2}$ and $\mu(t_{2j-1}) = 2 - \frac{1}{(2j+1)^2}$, then the equation $y^\Delta = -y$ is not Ulam-Hyers stable.

These (and also other) examples shows that the Ulam-Hyers stability of linear dynamical equations with constant coefficients on a timescale $\mathbb{T}$ is closely related to the behavior of the exponential functions defined on $\mathbb{T}$. Moreover, this behavior is connected also with the inner structure of the timescale, not only with the constants (or functions) defining the exponential functions. For this reason we formulate our results in terms of the asymptotical behavior of the exponential functions and by some remarks we emphasize special classes of constants, for which we have Ulam-Hyers stability. Let $a \in \mathbb{C}$ be a complex number and $\mathbb{T}$ a time scale. Consider the following conditions:

S1 $|e_a(t, t_0)|$ and $\int_{t_0}^t |e_a(t, \sigma(s))|\Delta s$ are bounded on $[t_0, \infty)_\mathbb{T}$;

S2 $\lim\limits_{t\to\infty} |e_a(t, t_0)| = \infty$ and $\int_t^\infty |e_a(t_0, \sigma(s))|\Delta s < \infty$, for all $t \in [t_0, \infty)_\mathbb{T}$;

S3 $|e_a(t, t_0)|$ is bounded on $[t_0, \infty)_\mathbb{T}$ and $\lim\limits_{t\to\infty} \int_{t_0}^t |e_a(s, t_0)|\Delta s = \infty$.

Remark 3.1.4. **a)** If $\mathbb{T} = \mathbb{R}$, and $|a| \neq 0$, one of the conditions S1 and S2 holds.

b) If $\mathbb{T} = \mathbb{Z}$ and $a \notin \{-2, 0\}$, one of the conditions S1 and S2 holds.

c) If $t_0 = 1$, $\mu(t_{2j}) = \frac{1}{(2j+2)^2}$ and $\mu(t_{2j-1}) = 2 - \frac{1}{(2j+1)^2}$, then the exponential function $e_{-1}(t, t_0)$ changes sign on each interval $[t_j, t_{j+1}]$ and condition S3 holds.

d) There are timescales $\mathbb{T}$ for which the modulus of some exponential functions has arbitrary large and arbitrary small values on each interval $[t, \infty)$. In such a case none of the previous conditions hold.

Theorem 3.1.5. *Consider the following dynamic equation:*

$$\left\{ \; y^\Delta(t) = ay(t) \right. \tag{3.1.1}$$

If S1 or S2 holds, then the above equation is Ulam-Hyers stable on $[t_0, +\infty)_\mathbb{T}$. *The same property is valid also for the inhomogeneous equation.*

Theorem 3.1.6. *Consider the following* n^{th} *order dynamic equation:*

$$\left\{ \; y^{\Delta(n)} - \sum_{k=1}^{n} a_k y^{\Delta(k)} = 0 \right. \tag{3.1.2}$$

Denote by $\lambda_1, \lambda_2, \ldots \lambda_k$ *the roots of the characteristic equation*

$$r^n - \sum_{k=1}^{n} a_k r^{n-k} = 0.$$

If $1 + \mu(t)\lambda_j \neq 0, \forall t \in \mathbb{T}$ *for all* $1 \leq j \leq n$, *and for each* λ_j *S1 or S2 is verified, then equation (3.1.2) is Ulam-Hyers stable on* $[t_0, +\infty)_\mathbb{T}$.

Remark 3.1.7. If $a > 0$, the exponential function $e_a(t, t_0)$ is positive, hence the integrals in S1, S2 and S3 can be calculated effectively. In this case equation 3.1.1 is always Ulam-Hyers stable. If $a = 0$, the equation $y^\Delta(t) = 0$ has only constant solution, while the perturbed equation $y^\Delta(t) = \varepsilon$ has the solution $y(t) = y(t_0) + \varepsilon(t - t_0)$, hence the equation 3.1.1 is not Ulam-Hyers stable. The same example shows that if at least one of the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ is 0, then the equation 3.1.2 is not Ulam-Hyers stable. The condition $1 + \mu(t)a \neq 0$ (and $1 + \mu(t)\lambda_j \neq 0$, $1 \leq j \leq n$) is necessary for the existence of the corresponding exponential function(s).

Remark 3.1.8. The previous theorems are an extension (a unified formulation) of the results from [31], [16], [26] and [32]. In [32] the critical value seems to be 1, but this is only because the equation is in the form $y_{n+1} = ay_n$ which is $y^\Delta = (a-1)y$.

Proof of theorem 3.1.5. The solution of the equation 3.1.1 is

$$y(t) = y_0 e_a(t, t_0). \tag{3.1.3}$$

The solution of the perturbed equation

$$z^\Delta(t) = az(t) + h(t), \tag{3.1.4}$$

can be represented as

$$z(t) = z_0 e_a(t, t_0) + \int_{t_0}^{t} h(s) e_a(t, \sigma(s)) \Delta s. \tag{3.1.5}$$

We need to estimate the difference between $y(t)$ and $z(t)$, if $|h(t)| < \varepsilon$, $\forall t \in \mathbb{T}$.

Case 1. If S1 holds, we have

$$|z(t) - y(t)| = \left| (z_0 - y_0) e_a(t, t_0) + \int_{t_0}^{t} h(s) e_a(t, \sigma(s)) \Delta s \right| \leq$$

$$\leq |(z_0 - y_0)| \cdot |e_a(t, t_0)| + \varepsilon \int_{t_0}^{t} |e_a(t, \sigma(s))| \Delta s.$$

These inequalities imply that if $M_1 > 0$ is an upper bound for $|e_a(t, t_0)|$ and $M_2 > 0$ an upper bound for $\int_{t_0}^{t} |e_a(t, \sigma(s))\Delta s|$, then by choosing y_0 such that $|y_0 - z_0| < \varepsilon$, we have $|y(t) - z(t)| < \varepsilon(M_1 + M_2)$, for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Remark 3.1.9. If $\mathbb{T} = \mathbb{R}$ and $a < 0$, this situation occurs.

Case 2. If S2 holds $a \neq 0$ and we have

$$|z(t) - y(t)| = \left| (z_0 - y_0) e_a(t, t_0) + \int_{t_0}^{t} h(s) e_a(t, \sigma(s)) \Delta s \right| \leq$$

$$\leq |e_a(t, t_0)| \left| z_0 - y_0 + \int_{t_0}^{t} h(s) e_a(t_0, \sigma(s)) \Delta s \right|$$

From condition S2 we deduce that the improper integral $\int_{t_0}^{\infty} h(s)e_a(t_0,\sigma(s))\Delta s$ is absolutely convergent, hence it is also convergent. If we choose

$$y_0 = z_0 + \int_{t_0}^{\infty} h(s)e_a(t_0,\sigma(s))\Delta s,$$

we have

$$|z(t)-y(t)| \le \varepsilon \left| e_a(t,t_0)\cdot \int_t^{\infty} e_a(t_0,\sigma(s))\Delta s \right|.$$

But

$$\int_t^{\infty} e_a(t_0,\sigma(s))\Delta s = -\frac{1}{a}\int_t^{\infty} \frac{-a}{(1+\mu(s)a)e_a(s,t_0)}\Delta s =$$

$$= -\frac{1}{a}\left[\lim_{s\to\infty}\frac{1}{e_a(s,t_0)} - \frac{1}{e_a(t,t_0)}\right] = \frac{1}{ae_a(t,t_0)},$$

so

$$|z(t)-y(t)| \le \frac{\varepsilon}{|a|},\ \forall t \in [t_0,\infty)$$

and the equation 3.1.1 is Ulam-Hyers stable.

Remark 3.1.10. If $\mathbb{T}=\mathbb{R}$ and $a>0$, or $\mathbb{T}=\mathbb{Z}$ and $a>1$ this situation occurs.

□

Remark 3.1.11. If we consider the inhomogeneous equation $y^{\Delta} = ay+f$ and the perturbed equation $z^{\Delta} = az+f+h$, the difference of the solutions is not depending on f, so the Ulam-Hyers stability automatically is transferred to the inhomogeneous equation.

Proof of the theorem 3.1.6. If y is a solution of the equation

$$y^{\Delta(n)} - \sum_{k=1}^{n} a_k y^{\Delta(k)} = h(t)$$

and λ_n is a root of the corresponding characteristic equation, then the function $z = y' - \lambda_n \cdot y$ is satisfying an $(n-1)^{th}$ order inhomogeneous equation with constant coefficients for which the roots of the characteristic equations are $\lambda_1,\dots,\lambda_{n-1}$ and the inhomogeneity is the same. Hence by an inductive argument there exists a solution y_1 of the corresponding homogeneous equation and a constant c_1, such that $|z(t)-y_1(t)| < c_1\cdot\varepsilon$, $\forall t\in[t_0,\infty)$. Applying Theorem 3.1.5 for the inhomogeneous equation $y' - \lambda_n y = y_1$, we deduce the existence of a function y_2 with the properties $y_2^{\Delta} = \lambda_n y_2 + y_1$ and $|y-y_2| < c_2\cdot c_1\varepsilon$, $\forall t\in [t_0,\infty)$. But y_1 being the solution of the $(n-1)^{th}$ order equation y_2 is the solution of the initial n^{th} order equation, so we have the Ulam-Hyers stability of the initial equation. □

3.2 Ulam-Hyers stability of some integral equations

In this section we study the Ulam-Hyers stability of the integral equation

$$u(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1,t_2)u(s_1,t_2)\Delta_1 s_1, \tag{3.2.1}$$

and of the more general equation

$$\begin{aligned} u(t) = w(t) + \int_a^t a_1(s_1)u(s_1)\Delta s_1 + \\ + \int_a^t \int_a^{s_1} a_2(s_2)u(s_2)\Delta s_2 \Delta s_1 + \cdots + \int_a^t \int_a^{s_1} \dots \int_a^{s_{n-1}} a_n(s_n)u(s_n)\Delta s_n \dots \Delta s_1. \end{aligned}$$

Equations of this types appear when we transform higher order dynamic equations to fixed point problems, hence the Ulam-Hyers stability of these equations also provides information on the Ulam-Hyers stability of the dynamic equations. The main difference consists in the fact that using integral equations, the Ulam-Hyers stability is obtained in a well chosen metric space, neither the conditions, nor the conclusions of these theorems are not the same as in the classical framework (the closeness of the approximative solution and the exact solution is not measured in the classical sense).

We assume, that the reader is already familiar with all the notions about the time scale calculus recalled in the previous chapter. We also will use the Bielecki type metrics introduced in the previous chapter.

For the simplicity of notation in what follows we use again the following notations: $X := C([a_1,\sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2,\sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R})$, $D_1 := [a_1,\sigma_1(b_1)]_{\mathbb{T}_1}$, $D_2 := [a_2,\sigma_2(b_2)]_{\mathbb{T}_2}$.

Using the Lemma 0.2.5 we obtain the following results:

Theorem 3.2.1. *Let $w, a, b \in X$, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$. Then the integral equation*

$$u(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1,t_2)u(s_1,t_2)\Delta_1 s_1, \tag{3.2.2}$$

is Ulam-Hyers stable on $D_1 \times D_2$.

Proof. Using the notations from the proof of Theorem 2.2.4, we have $M_1 < \infty$ and $M_2 < \infty$, so the operator defined as

$$A(u)(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1,t_2)u(s_1,t_2)\Delta_1 s_1, \tag{3.2.3}$$

is a contraction with the positive constant $q := \frac{M_1+\beta M_2}{\alpha\beta}$, if we choose α, β such that $q < 1$. From the Lemma 0.2.5 we deduce that our operator A is a c-weakly PO with the positive constant $c_A = \frac{\alpha\beta}{\alpha\beta - M_1 - \beta M_2}$ and from Theorem 0.2.3 we obtain the Ulam-Hyers stability of the equation (3.2.2). □

Using a similar argument we obtain the Ulam-Hyers stability of a more general integral equation.

Theorem 3.2.2. *Let $w, a, b \in X$, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$. Further let be the functions $f, g \in C(D_1 \times D_2 \times \mathbb{R}, \mathbb{R})$ with a Lipschitz property in their last variables. Then the integral equation*

$$\begin{aligned} u(t_1, t_2) = w(t_1, t_2) + &\int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) f(s_1, s_2, u(s_1, s_2)) \Delta_1 s_1 \Delta_2 s_2 \\ &+ \int_{a_1}^{t_1} b(s_1, t_2) g(s_1, t_2, u(s_1, t_2)) \Delta_1 s_1, \end{aligned}$$

is Ulam-Hyers stable on $D_1 \times D_2$.

As a consequence we give the following results for second order dynamic equations with constant coefficients.

Theorem 3.2.3. *For the real constants c_1 and c_2 we consider the second-order linear dynamic equation*

$$x^{\Delta\Delta}(t) + c_1 x^{\Delta}(t) + c_2 x(t) = 0, \tag{3.2.4}$$

on a time scale interval $[a, b]_{\mathbb{T}}$. This equation is always Ulam-Hyers stable on $[a, b]_{\mathbb{T}}$.

Proof. We integrate the equation (3.2.4) from a to t and we have:

$$x^{\Delta}(t) - x^{\Delta}(a) + c_1(x(t) - x(a)) + c_2 \int_a^t x(s) \Delta s = 0.$$

Now we integrate once again this equation from a to t, and renaming the variables we get:

$$\begin{aligned} x(t) = &x(a) - \left(x^{\Delta}(a) + c_1 x(a)\right) a + \left(x^{\Delta}(a) + c_1 x(a)\right) t - \\ &- c_2 \int_a^t \int_a^s x(\xi) \Delta\xi \Delta s - c_1 \int_a^t x(s) \Delta s \end{aligned}$$

Using the function $w(t) := x(a) - \left(x^{\Delta}(a) + c_1 x(a)\right) a + \left(x^{\Delta}(a) + c_1 x(a)\right) t$, we define the operator on $C[a, b]_{\mathbb{T}}$

$$A(x)(t) = w(t) - c_2 \int_a^t \int_a^s x(\xi) \Delta\xi \Delta s - c_1 \int_a^t x(s) \Delta s. \tag{3.2.5}$$

Equation 3.2.4 is equivalent to the equation $Ax = x$, so theorem 3.2.1 can be applied to obtain the Ulam-Hyers stability of the fixed point equation derived from A. Due to the equivalent transformations of (3.2.4), the boundedness of the interval and the boundedness of the exponential functions we also have the Ulam-Hyers stability of the dynamical equation 3.2.4. ∎

Theorem 3.2.4. *Let $p, q, f \in C_{rd}[a,b]_{\mathbb{T}}$ and consider the following second-order inhomogeneous delta dynamic equation with variable coefficients:*

$$x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) + q(t)x(t) = f(t),\ t \in [a,b]_{\mathbb{T}}. \tag{3.2.6}$$

If p is Δ differentiable and $p = p^{\sigma}$ on its domain, then the dynamic equation (3.2.6) *is Ulam-Hyers stable.*

Proof. We use the same idea as in the proof of Theorem 3.2.3, what is, we want to construct an integral operator and we prove that it is c-weakly PO.

So we integrate 3.2.6 from a to t:

$$x^{\Delta}(t) - x^{\Delta}(a) + \int_a^t p(s)x^{\Delta}(s)\Delta s + \int_a^t q(s)x(s)\Delta s = \int_a^t f(s)\Delta s.$$

Furthermore with partial integration using that $p = p^{\sigma}$ we have:

$$x^{\Delta}(t) - x^{\Delta}(a) + p(t)x(t) - p(a)x(a) - \int_a^t p^{\Delta}(s)x(s)\Delta s + \int_a^t q(s)x(s)\Delta s = \int_a^t f(s)\Delta s.$$

Integrating once more from a to t and arrange the terms we get:

$$\begin{aligned} x(t) =& x(a) + \left(x^{\Delta}(a) + p(a)x(a)\right)(t-a) + \int_a^t \int_a^s f(\xi)\Delta\xi\Delta s \\ &+ \int_a^t \int_a^s \left(p^{\Delta}(\xi) - q(\xi)\right) x(\xi)\Delta\xi\Delta s - \int_a^t p(s)x(s)\Delta s \end{aligned}$$

So with $w(t) := x(a) + \left(x^{\Delta}(a) + p(a)x(a)\right)(t-a) + \int_a^t \int_a^s f(\xi)\Delta\xi\Delta s$ we define the operator A on $C[a,b]_{\mathbb{T}}$ as it follows:

$$A(x)(t) := w(t) + \int_a^t \int_a^s \left(p^{\Delta}(\xi) - q(\xi)\right) x(\xi)\Delta\xi\Delta s - \int_a^t p(s)x(s)\Delta s.$$

By applying Theorem 3.2.1 we obtain the Ulam-Hyers stability of the fixed point equation generated by A. Due to the boundedness of the interval and of the exponential functions we also obtain the Ulam-Hyers stability of the equation (3.2.6). □

Remark 3.2.5. Theorem 3.2.3 and Theorem 3.2.4 can be obtained also without using Theorem 3.2.1 by using directly 0.2.5 and a metric space with functions having only one variable (see the proof of Theorem 3.2.6). These results are more general than Theorem 1.5. in [6].

Theorem 3.2.3 and Theorem 3.2.4 can also be generalized in order to imply the Ulam-Hyers stability of linear delta dynamic equations of order n.

Theorem 3.2.6. *Let $w, a_1, a_2, \ldots, a_n \in C[a, \sigma(b)]_{\mathbb{T}}$, $\sigma(b) < \infty$. Then the integral equation*

$$\begin{aligned} u(t) = w(t) + \int_a^t a_1(s_1)u(s_1)\Delta s_1 + \int_a^t \int_a^{s_1} a_2(s_2)u(s_2)\Delta s_2 \Delta s_1 \\ + \cdots + \int_a^t \int_a^{s_1} \ldots \int_a^{s_{n-1}} a_n(s_n)u(s_n)\Delta s_n \ldots \Delta s_1, \end{aligned}$$

is Ulam-Hyers stable on $[a, \sigma(b)]_{\mathbb{T}}$.

Proof. In the proof we use the same idea as in the proof of the Theorem 3.2.1, based on a one dimensional Bielecki type metric and norm in order to prove the contractive property of the defined operator. Let us define the integral operator $A : C[a, \sigma(b)]_{\mathbb{T}} \to C[a, \sigma(b)]_{\mathbb{T}}$, by

$$\begin{aligned} A(u)(t) = w(t) + \int_a^t a_1(s_1)u(s_1)\Delta s_1 + \int_a^t \int_a^{s_1} a_2(s_2)u(s_2)\Delta s_2 \Delta s_1 \\ + \dots + \int_a^t \int_a^{s_1} \dots \int_a^{s_{n-1}} a_n(s_n)u(s_n)\Delta s_n \dots \Delta s_1, \end{aligned}$$

The functions $a_1, a_2, \dots, a_n \in C[a, \sigma(b)]_{\mathbb{T}}$, so there exist positive real constants $M_1 < \infty, \dots, M_n < \infty$ such that $|a_1(t)| < M_1, \dots, |a_n(t)| < M_n$, for all $t \in [a, \sigma(b)]_{\mathbb{T}}$. If $u, v \in C[a, \sigma(b)]_{\mathbb{T}}$, we have

$$\begin{aligned} |A(u)(t) - A(v)(t)| \leq & M_1 \int_a^t |u(s_1) - v(s_1)|\Delta s_1 + \\ & + M_2 \int_a^t \int_a^{s_1} |u(s_2) - v(s_2)|\Delta s_2 \Delta s_1 + \dots + \\ & + M_n \int_a^t \int_a^{s_1} \dots \int_a^{s_{n-1}} |u(s_n) - v(s_n)|\Delta s_n \dots \Delta s_1. \end{aligned}$$

$$\begin{aligned} & \leq M_1 ||u - v||_\alpha \frac{e_\alpha(t, a)}{\alpha} + M_2 ||u - v||_\alpha \frac{e_\alpha(t, a)}{\alpha^2} + \dots + M_n ||u - v||_\alpha \frac{e_\alpha(t, a)}{\alpha^n} \\ & = ||u - v||_\alpha e_\alpha(t, a) \left(\frac{M_1}{\alpha} + \frac{M_2}{\alpha^2} + \dots + \frac{M_n}{\alpha^n} \right). \end{aligned}$$

Now we divide the inequality by the positive function $e_\alpha(t, a)$ and taking the supremum over $t \in [a, \sigma(b)]_{\mathbb{T}}$ we have

$$||A(u) - A(v)||_\alpha \leq ||u - v||_\alpha \left(\frac{M_1}{\alpha} + \frac{M_2}{\alpha^2} + \dots + \frac{M_n}{\alpha^n} \right) \tag{3.2.7}$$

Let $M := \max\{M_1, M_2, \dots, M_n\}$ and so we get

$$||A(u) - A(v)||_\alpha \leq ||u - v||_\alpha M \frac{1}{\alpha} \frac{1 - \frac{1}{\alpha^n}}{1 - \frac{1}{\alpha}} \leq ||u - v||_\alpha \frac{M}{\alpha - 1}. \tag{3.2.8}$$

If $\frac{M}{\alpha-1} < 1$, the operator A is a contraction, and due to Lemma 0.2.5 we have the c-weakly PO property of A. Moreover we also obtain the Ulam-Hyers stability of the integral equation and this implies the Ulam-Hyers stability of the n^{th} order dynamic equation with constant coefficients (because of the boundedness of the interval and of the exponential function $e_a(t, a)$). ☐

We can generalize the Theorem 3.2.6 in the following way:

Theorem 3.2.7. *Let $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_n$ be arbitrary time scales and*

$$[a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \subset \mathbb{T}_1, \dots, [a_n, \sigma_n(b_n)]_{\mathbb{T}_n} \subset \mathbb{T}_n$$

time scale intervals such that $\sigma_1(b_1) < \infty, \dots, \sigma_n(b_n) < \infty$. Denote $Y := C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times \dots \times [a_n, \sigma_n(b_n)]_{\mathbb{T}_n}, \mathbb{R})$.

If $w, f_1, \dots, f_n \in Y$, the operator $A : Y \to Y$ defined by

$$\begin{aligned} A(u)(t_1, \dots, t_n) = w(t_1, \dots, t_n) + \int_{a_1}^{t_1} f_1(s_1, t_2, \dots, t_n) u(s_1, t_2, \dots, t_n) \Delta_1 s_1 + \\ + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f_2(s_1, s_2, t_3, \dots, t_n) u(s_1, s_2, t_3, \dots, t_n) \Delta_2 s_2 \Delta_1 s_1 + \dots + \\ + \int_{a_1}^{t_1} \int_{a_2}^{t_2} \dots \int_{a_n}^{t_n} f_n(s_1, \dots, s_n) u(s_1, \dots, s_n) \Delta_n s_n \dots \Delta_1 s_1 \end{aligned}$$

is a c-weakly PO, moreover the fixed point equation $u = A(u)$ is Ulam-Hyers stable on $D_1 \times \dots \times D_n$, where $D_i = [a_i, \sigma_i(b_i)]_{\mathbb{T}_i}, i = 1, \dots, n$.

The proof of this theorem uses a similar argument as the proof of Theorem 3.2.3 and is based on the norm

$$\|u\|_{\alpha_1, \dots, \alpha_n} = \sup_{s_1 \in D_1, \dots, s_n \in D_n} \frac{\|u(s_1, \dots, s_n)\|}{e_{\alpha_1}(s_1, a_1) \dots e_{\alpha_n}(s_n, a_n)} \tag{3.2.9}$$

for all $u \in Y$. For this reason we omit the details.

3.3 Ulam-Hyers stability of linear delta dynamic systems

Here we study the Ulam-Hyers stability of linear delta dynamic systems

$$u^{\Delta}(t) = K(t)u(t) + F(t), \tag{3.3.1}$$

where $u(t) = (u_1(t), \dots, u_n(t))^T$, with $u_1, \dots, u_n : D \to \mathbb{R}$, $K(t) = (k_{ij}(t))_{i,j=1,\dots,n}$ is a matrix of dimensions $n \times n$, $u^{\Delta}(t) = \left(u_1^{\Delta}(t), \dots, u_n^{\Delta}(t)\right)^T$, $f_1, \dots, f_n : D \to \mathbb{R}$ and $F(t) = (f_1(t), \dots, f_n(t))^T$, on an arbitrary time scale interval $D := [a, \sigma(b)]_{\mathbb{T}}$. We use the one dimensional version of the Bielecki type metric (2.2.5) for the function space $C(D, \mathbb{R}^n)$. With this metric d_α the space $X = (C(D, \mathbb{R}^n), d_\alpha)$ is a Banach space.

Theorem 3.3.1. *If the functions $k_{ij}, f_i \in C_{rd}(D, \mathbb{R})$, $\forall i, j = 1, \dots, n$, then the equation (3.3.1) is Ulam-Hyers stable on D.*

Proof. Without loss of generality we can assume, that $u(a) = 0$. Integrating the equation (3.3.1) from a to t we have

$$u(t) = \int_a^t K(s)u(s)\Delta s + \int_a^t F(s)\Delta s. \tag{3.3.2}$$

If we define the operator $A : C(D, \mathbb{R}^n) \to C(D, \mathbb{R}^n)$ by

$$A(u)(t) := \int_a^t F(s)\Delta s + \int_a^t K(s)u(s)\Delta s, \tag{3.3.3}$$

we need to prove, that there exists a positive constant α such that A is a contraction on X. $k_{ij} \in C_{rd}(D, \mathbb{R}^n)$ implies that there exists a positive constant $M < \infty$, such that $||K(t)|| \leq M, \forall t \in D$, where $|| \cdot ||$ is a matrix norm. If $u, v \in C(D, \mathbb{R}^n)$, we have

$$\begin{aligned} ||A(u)(t) - A(v)(t)|| &\leq \int_a^t ||K(s)(u(s) - v(s))||\Delta s \leq \int_a^t ||K(s)||||u(s) - v(s)||\Delta s \\ &\leq M \int_a^t \frac{||u(s) - v(s)||}{e_\alpha(s, a)} e_\alpha(s, a)\Delta s \\ &\leq \frac{M}{\alpha}||u - v||_\alpha e_\alpha(t, a) \end{aligned}$$

Dividing the inequality by the positive function $e_\alpha(t, a)$, and taking the supremum over $t \in D$ we have

$$||A(u) - A(v)||_\alpha \leq \frac{M}{\alpha}||u - v||_\alpha.$$

If $\frac{M}{\alpha} < 1$, the operator A is a contraction, so by the Lemma 0.2.5 and Theorem 0.2.3 we deduce the Ulam-Hyers stability of the fixed point equation $u = A(u)$. Due to the equivalent transformations, and the boundedness of the interval we also have the Ulam-Hyers stability of the equation (3.3.1). $\square$

Remark 3.3.2. The previous results show that on a bounded intervals the Ulam-Hyers stability can be proved using a unified approach. On the other hand these results may not be relevant on some timescales (such as $\mathbb{Z}$). The theory of Picard operator can also be applied on unbounded intervals, where the generated function spaces are gauge spaces, but in generally we can establish only generalized Ulam-Hyers-Rassias stability and this is not the aim of this thesis.

Chapter 4

Ulam-Hyers stability of elliptic PDEs in Sobolev spaces

In this chapter we would like to study the Ulam-Hyers stability of some linear and nonlinear elliptic PDEs, using eventually some tools from the previous chapters.

Later on we always assume, that we are working in bounded, open, connected domains of $\mathbb{R}^d, d \geq 2$, with Lipschitz boundary.

We will see that in the case of the linear problems, the Ulam-Hyers stability does not say much in plus, because it will be the consequence of some usual elliptic estimations, using the Sobolev embedings, and some other usual inequalities as Poincaré's and Cauchy-Schwartz inequalities. While in the case of nonlinear problems the question of Ulam-Hyers stability will be not so trivial.

But at first for the coherency of the thesis, we will present some well known results about Lebesgue and Sobolev spaces and other tools, we will need later on.

4.1 Preliminaries from functional analysis

As we said before, in what it follows let $\Omega \subset \mathbb{R}^d, d \geq 2$ a bounded, connected, open subset of $\mathbb{R}^d$ with Lipschitz boundary.

For a real number $p \in [1, +\infty)$ we define the Lebesgue space $L^p(\Omega)$ as it follows

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \middle| \ u \text{ is mesurable and } \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}} < +\infty \right\}$$

For $p = +\infty$, we have

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \middle| \ u \text{ is mesurable and} \sup_{x \in \overline{\Omega}} |u(x)| < +\infty \right\}$$

On these spaces we define the norms

$$||u||_{L^p} = \begin{cases} \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \sup_{x \in \overline{\Omega}} |u(x)|, & \text{if } p = +\infty. \end{cases}$$

Remark 4.1.1. We remark, that actually the above defined Lebesgue spaces contain equivalence classes, where two functions are equivalent, if the set where they do not coincide has Lebesgue measure zero in $\mathbb{R}^d$.

It is also well known, that $(L^p(\Omega), ||\cdot||_{L^p})$ are Banach spaces for all $p = [1, +\infty]$.

Remark 4.1.2. Previously we were used, and further we are using the integrations respect to the Lebesgue measure on $\mathbb{R}^d$, more precisely we denote by $dx = d\mathcal{L}^d(x)$, the d-dimensional Lebesgue measure. This is natural of course.

We similarly define the Sobolev spaces $W^{k,p}(\Omega), W_0^{k,p}(\Omega), H^k(\Omega), H_0^k(\Omega)$ as it follows.

$$W^{k,p} = \left\{ u \in L^p(\Omega) \middle| D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k \right\}$$

Here α is a positive multi index and for a function $u \in L^p(\Omega)$ we denote by $D^\alpha u$ its derivative of order α in the sense of distributions. The "power" k is a positive integer and the $p \in [1, +\infty]$.

We define a norm on the space $W^{k,p}(\Omega)$ as

$$||u||_{W^{k,p}} = \begin{cases} \left(\sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p}^p \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty) \\ \sum_{|\alpha| \leq k} ||D^\alpha||_{L^\infty}, & \text{if } p = +\infty \end{cases}$$

We can define the space $W_0^{k,p}(\Omega)$ as the closure of the space of smooth functions with compact support, $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to the $||\cdot||_{W^{k,p}}$ norm. This definition of $W_0^{k,p}(\Omega)$ is equivalent to the space of functions from $W^{k,p}(\Omega)$ which are zero on $\partial\Omega$ in the trace sense.

By definition $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

We denote by $W^{-k,p'}(\Omega) = \left(W_0^{k,p}(\Omega) \right)^*$, the topological dual space of $W_0^{k,p}(\Omega)$, where p' is a conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Clearly we have $H^{-k}(\Omega) = \left(H^k(\Omega) \right)^*$.

For a Banach space X and its topological dual X^*, we denote the duality pairing between X^* and X by $< \cdot, \cdot >_{X^* \times X}$.

Remark 4.1.3. It is well known, that all the above defined spaces endowed with their norms are Banach spaces, and the spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are Hilbert spaces. For finite p we also have, that the spaces $W^{k,p}(\Omega)$ are separable and if in addition $p > 1$ they are also reflexive and uniformly convex.

For a very detailed study of the Sobolev spaces we recommend the book of Adams and Fournier ([2]) and the book of H. Brezis ([15]).

We can also define Sobolev spaces with non-integer "power" k on the whole space $\mathbb{R}^d$, with the help of Fourier transformation. An equivalent construction of these, so called Besov spaces on bounded and unbounded domains can be done by interpolation methods between Banach spaces. The very detailed construction of the Besov spaces can also be found in the book of Adams, ([2]).

We present here some important well known theorems without proofs, which we will use later on.

Theorem 4.1.4 (Fredholm Alternative)**.** *Let H be a Hilbert space and $K : H \to H$ a compact linear operator. Then the following alternative holds:*

Either

(a) $I + K$ is an isomorphism of H, that is for any $v \in H$ there is a unique $u \in H$, such that $u + Ku = v$ and there is a $c > 0$ independent of v such that $||u||_H \leq c||v||_H$,

or

(b) $\ker(I + K) \neq \{0\}$, that is, there is $u \in H \setminus \{0\}$, such that $u + Ku = 0$.

Here we denoted $I : H \to H$, the identity operator on H.

Theorem 4.1.5 (Refined version of the Fredholm Alternative)**.**

$$R(I + K) = (\ker(I + K^*))^{\perp},$$

where $R(I + K)$ denotes the range of the operator $I + K$, while K^* the adjoint operator of K.

In other words, in case (a) the equation $u + Ku = v$ has a solution if and only if $\forall h \in \ker(I + K^*)$, we have $(h, v)_H = 0$.

Moreover, $\ker(I + K)$ has the same dimension as $\ker(I + K^*)$ so, in case (b) if v satisfies $\forall h \in \ker(I + K^*) : (v, h)_H = 0$, then the set of solutions u of $u + Ku = v$ is an affine subspace of H of dimension $dim \ker(I + K) = dim \ker(I + K^*)$.

Remark 4.1.6. We remark, that in the special case of $H_0^1(\Omega)$, we can define a norm, which is equivalent to the restriction of $|| \cdot ||_{H^1}$ norm to $H_0^1(\Omega)$, because of Poincaré's inequality. So this norm is:

$$||u||_{H_0^1} = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}.$$

Now let us define, what does it mean the Ulam-Hyers stability of an elliptic equation. Let us take for example Poisson's problem with homogeneous boundary conditions. We will give the rigorous assumptions for the function space, domain, right hand side, etc. later on.

Definition 4.1.7. Let us consider the problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1.1}$$

We say that the above equation is Ulam-Hyers stable, if there exists a positive constant C such that for every $\varepsilon > 0$ and for each solution v of the problem:

$$\begin{cases} -\Delta v = \bar{f}, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1.2}$$

where $||f - \bar{f}|| \leq \varepsilon$, there exist a solution u of the original problem (4.1.1), such that

$$||u - v|| \leq C \cdot \varepsilon.$$

Remark 4.1.8. We can use this type of definition to define the Ulam-Hyers stability of any type of elliptic PDE.

Remark 4.1.9. We remark, that a well-known natural method in PDEs is to work with some kind of week solutions, after when we got some results, concerning the existence, uniqueness, etc. of these week solutions, with some regularity assumptions, we will have information about the classical solutions. This is the situation also in case of Ulam-Hyers stability. We will get at first results for week solutions, then with some regularity argumentations these will hold also for classical solutions.

4.2 Linear problems

At first we study the stability in the Ulam-Hyers sense of some particular linear elliptic PDEs, namely Poisson's equation with Dirichlet and Neumann boundary conditions, and then we will give more general results.

If we don't say otherwise, we are using the assumptions mentioned in the previous subsection.

Theorem 4.2.1. *Let $f \in H^{-1}(\Omega)$. Then Poisson's problem with Dirichlet boundary conditions*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{4.2.1}$$

is Ulam-Hyers stable with respect to weak solutions in $H_0^1(\Omega)$.

We can prove this theorem in two ways. We can work with the abstract method of Picard operators, or simply we can use just some elliptic estimations. However the second approach is more simpler, we will give the other proof in details too, because we can use this idea also in the nonlinear case, or in cases, where the second method does not work.

First proof, with fixed point approach. The idea of the proof is to construct a fixed point equation as (0.2.1), then try to prove, that the operator is c-weakly Picard, finally we can conclude by the Theorem 0.2.3 the Ulam-Hyers stability of the desired problem.

At first we remark, that it is well known due to Riesz's representation theorem, that the problem (4.2.1) has a unique weak solution in $H_0^1(\Omega)$.

We define an abstract operator as follows. Let $A : H_0^1(\Omega) \to H_0^1(\Omega)$, which associates to an input $v \in H_0^1(\Omega)$ the unique weak solution of the modified problem

$$\begin{cases} -\Delta u + \lambda u = f + \lambda v, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{4.2.2}$$

where $\lambda > 0$ is a positive real constant to be chosen small enough. We will give the precise assumptions to the λ later.

By Riesz's representation theorem it is obvious again, that for any input $v \in H_0^1(\Omega)$ the problem (4.2.2) has a unique weak solution in $H_0^1(\Omega)$, due to the fact that the right hand side is an element in $H^{-1}(\Omega)$ (because for bounded domains, we can use a Sobolev embedding theorem, since there exists a continuous and compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $L^2(\Omega) \subset H^{-1}(\Omega)$.) Hence the operator A is well defined.

Next we are going to prove, that A is a c-weakly Picard operator. Actually we can prove a stronger fact, namely that it is a contraction.

Let us write the weak formulation of the problem (4.2.2) and prove the claim. The problem (4.2.2) is equivalent to

$$\int_\Omega \nabla u(x)\cdot\nabla\varphi(x)dx+\lambda\int_\Omega u(x)\varphi(x)dx =< f,\varphi >_{H^{-1}(\Omega)\times H_0^1(\Omega)} +\lambda\int_\Omega v(x)\varphi(x)dx, \quad (4.2.3)$$

$\forall\varphi\in H_0^1(\Omega)$.

Now we take two input functions $v_1, v_2 \in H_0^1(\Omega)$, and by definition let $u_i = A(v_i), i = 1,2$ the unique weak solutions of (4.2.2). Our aim is to give an approximation to $||u_1 - u_2||_{H_0^1}$.

We write the corresponding weak formulations for v_1, u_1 and v_2, u_2 and take the difference. Without the loss of the generality we can use the same test function $\varphi\in H_0^1(\Omega)$ in both formulations. Hence we will have:

$$\int_\Omega \nabla(u_1-u_2)(x)\cdot\nabla\varphi(x)dx+\lambda\int_\Omega (u_1-u_2)(x)\varphi(x)dx=\lambda\int_\Omega (v_1-v_2)(x)\varphi(x)dx,$$

$\forall\varphi\in H_0^1(\Omega)$.

Now let $\varphi = u_1 - u_2$, so we have

$$\int_\Omega |\nabla(u_1-u_2)|^2(x)dx+\lambda\int_\Omega |u_1-u_2|^2(x)dx=\lambda\int_\Omega (v_1-v_2)(x)(u_1-u_2)(x)dx.$$

This implies, that

$$||u_1-u_2||^2_{H_0^1}\le\lambda\int_\Omega (v_1-v_2)(x)(u_1-u_2)(x)dx.$$

Using the Cauchy-Schwarz inequality for the right hand side, we have

$$||u_1-u_2||^2_{H_0^1}\le\lambda||v_1-v_2||_{L^2}||u_1-u_2||_{L^2}.$$

Using twice Poincaré's inequality we have

$$||u_1-u_2||^2_{H_0^1}\le\lambda\cdot C_\Omega\cdot C_\Omega||v_1-v_2||_{H_0^1}||u_1-u_2||_{H_0^1},$$

where $C_\Omega > 0$ is the positive Poincaré constant, depending only on the geometry of Ω, and its value is actually $C_\Omega = \frac{1}{\lambda_1}$, where $\lambda_1 > 0$ is the least eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

Now dividing the inequality by $||u_1 - u_2||_{H_0^1}$, we have

$$||A(v_1)-A(v_2)||_{H_0^1}=||u_1-u_2||_{H_0^1}\le\lambda\cdot C_\Omega^2||v_1-v_2||_{H_0^1}.$$

So if we choose $\lambda < \frac{1}{C_\Omega^2} = \lambda_1^2$, then the operator A is a contraction.

So by the Theorem 0.2.3 and Theorem 0.2.5 we have the Ulam-Hyers stability of (4.2.2), hence the Ulam-Hyers stability of (4.2.1). $\square$

Second proof using elliptic estimations. Let $\varepsilon > 0$ be fixed. We consider the problem

$$\begin{cases} -\Delta v = g, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.2.4}$$

where $g \in H^{-1}(\Omega)$ and $||f - g||_{H^{-1}} \leq \varepsilon$. We know that the solution v exists and is unique in $H_0^1(\Omega)$, so is for the original problem (4.2.1), let us call this u. We want to show, that there exists a positive constant $C > 0$, independent of ε, such that

$$||u - v||_{H_0^1} \leq C\varepsilon.$$

We subtract the two equations and write the weak formulation of this equation, then take as test function $u - v \in H_0^1(\Omega)$. So we have

$$\int_\Omega |\nabla(u - v)|^2 \leq < f - g, u - v >_{H^{-1}, H_0^1} .$$

Now using an estimation for the right hand side, we will have

$$||u - v||_{H_0^1} \leq \varepsilon.$$

So the original problem is Ulam-Hyers stable. □

Now we study the stability of Poisson's problem, with non homogeneous Dirichlet boundary condition. We have the following result:

Theorem 4.2.2. *Let $f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Then Poisson's problem with Dirichlet boundary conditions*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{4.2.5}$$

is Ulam-Hyers stable with respect to weak solutions in $H^1(\Omega)$.

Proof. If $\partial\Omega$ is Lipschitz, we know by the trace theorem, that for any $g \in H^{\frac{1}{2}}(\Omega)$ there exits a function $u_g \in H^1(\Omega)$, such that the trace of u_g is g.

On the other hand, we know by the maximum principle, that the problem (4.2.5) has at most one solution.

We pose $w = u - u_g$, and the problem (4.2.5) is equivalent to

$$\begin{cases} -\Delta w = f + \Delta u_g, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega \end{cases} \tag{4.2.6}$$

So we transformed our problem with non homogeneous Dirichlet boundary condition into one, with homogeneous boundary conditions.

Notice that, $u_g \in H^1(\Omega)$, so we do not have, that $\Delta u_g \in L^2(\Omega)$, but we have that $\Delta u_g \in H^{-1}(\Omega)$. So the right hand side is in $H^{-1}(\Omega)$.

We also remark here, that it is well known, that the original problem (4.2.5) has a unique weak solution, independent of the Dirichlet lift function u_g.

By the Theorem 4.2.1 we have the Ulam-Hyers stability of the problem (4.2.6) and by equivalent transformation, we will have the Ulam-Hyers stability of the original problem (4.2.5). □

Now we formulate some stability results for Poisson's problem with Neumann boundary conditions.

Theorem 4.2.3. *Let $f \in L^2(\Omega)$. Then the problem*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \tag{4.2.7}$$

is Ulam-Hyers stable with respect to weak solutions in $H^1(\Omega)/\mathbb{R}$ if and only if

$$\int_\Omega f(x)dx = 0. \tag{4.2.8}$$

Here $\frac{\partial u}{\partial n}$ denotes the derivative of u with respect to the outer normal vector n, and this is defined almost everywhere on $\partial\Omega$. The space $H^1(\Omega)/\mathbb{R}$ is the space of equivalence classes on $H^1(\Omega)$, where two functions are in the same class, if their difference is a constant.

Proof. We know from the Fredholm Alternative theorem, that the problem (4.2.7) has weak solutions if and only if the compatibility condition (4.2.8) holds. Also by this theorem we know, that the affine solution space of this problem is of the form $u_0 + c$, where $u_0 \in H^1(\Omega)$ is a particular solution and $c \in \mathbb{R}$ is an arbitrary constant. Hence the problem has a unique weak solution in the space $H^1(\Omega)/\mathbb{R}$.

We know that the above defined $||\cdot||_{H_0^1}$ norm is a norm on $H^1(\Omega)/\mathbb{R}$ (because of the Poincaré-Wirtinger inequality) and moreover $\left(H^1(\Omega)/\mathbb{R}, ||\cdot||_{H_0^1}\right)$ is a Banach space.

In order to prove an Ulam-Hyers stability result for this problem, we use the same technique, as in the proof of Theorem 4.2.1. So we modify the problem (4.2.7) into

$$\begin{cases} -\Delta u + \lambda u = f + \lambda v, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.2.9}$$

where $\lambda > 0$ is a small enough positive constant, we give more assumptions on it later on and $v \in H^1(\Omega)$.

We know from Riesz's representation theorem, that for every $v \in H^1(\Omega)$, the problem (4.2.9) has a unique weak solution in $H^1(\Omega)$. We also observe, that if u is the solution of (4.2.9) for input v, that for an arbitrary constant $c \in \mathbb{R}$, $u + c$ will be the unique solution of (4.2.9) for the input $v + c$. This observation allows us, to define an operator

$$A : H^1(\Omega)/\mathbb{R} \to H^1(\Omega)/\mathbb{R},$$

which for an input function $v \in H^1(\Omega)/\mathbb{R}$ corresponds the unique weak solution of (4.2.9) in the same space. Hence this operator is well defined.

If we prove, that A is a c-weakly PO, then we are done, similarly to the proof of the Theorem 4.2.1, i.e. we will have the Ulam-Hyers stability of the problem (4.2.7).

Here we also can prove a stronger fact, what is that the operator A is a contraction.

We write the weak formulation of the problem, what is $\forall \varphi \in H^1(\Omega)$:

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x)dx + \lambda \int_\Omega u(x)\varphi(x)dx = \lambda \int_\Omega v(x)\varphi(x)dx + \int_\Omega f(x)\varphi(x)dx$$

We are looking actually weak solution in the space $H^1(\Omega)/\mathbb{R}$ and the operator A also acts on this space, that is why we can take in the weak formulation test functions $\forall\varphi \in H^1(\Omega)/\mathbb{R}$.

Now we take two inputs $v_1, v_2 \in H^1(\Omega)/\mathbb{R}$ and let $u_i = A(v_i), i = 1, 2$. Without loss of generality we can take the same test function φ for both equations, then after substraction and taking $\varphi := u_1 - u_2$, we have

$$\int_\Omega |\nabla(u_1 - u_2)(x)|^2 dx + \lambda \int_\Omega (u_1 - u_2)^2(x)dx = \lambda \int_\Omega (v_1 - v_2)(x)(u_1 - u_2)(x)dx$$

Using Cauchy-Schwarz inequality to the right hand side of the equation we have:

$$||u_1 - u_2||_{H_0^1} \leq \lambda ||v_1 - v_2||_{L^2} ||u_1 - u_2||_{L^2}.$$

On the space $H^1(\Omega)/\mathbb{R}$ all the functions, which differ only by a constant are in the same equivalence class, so they are equal an this space. Hence we have, that

$$||u_1 - u_2||^2_{H_0^1} \leq \lambda \Big|\Big|v_1 - v_2 - \frac{1}{|\Omega|}\int_\Omega (v_1 - v_2)(x)dx\Big|\Big|_{L^2} \Big|\Big|u_1 - u_2 - \frac{1}{|\Omega|}\int_\Omega (u_1 - u_2)(x)dx\Big|\Big|_{L^2}.$$

More precisely, in the step, when we took the input functions v_1 and v_2 and associated the solutions u_1 and u_2, we change nothing if we take instead $v_i - \frac{1}{|\Omega|}\int_\Omega v_i(x)dx$ and $u_i - \frac{1}{|\Omega|}\int_\Omega u_i(x)dx$, $i = 1, 2$. And taking these functions, we get the above inequality. Here we denoted by $|\Omega|$ the d-dimensional Lebesgue measure of Ω.

Using the Poincaré-Wirtinger inequality, we have

$$||u_1 - u_2||^2_{H_0^1} \leq \lambda C_\Omega^2 ||v_1 - v_2||_{H_0^1} ||u_1 - u_2||_{H_0^1},$$

so we have

$$||u_1 - u_2||_{H_0^1} \leq \lambda C_\Omega^2 ||v_1 - v_2||_{H_0^1},$$

where C_Ω is the Poincaré constant, depending only on the geometry of Ω.

And if $\lambda < \frac{1}{C_\Omega^2}$, then the operator A is a contraction, form where it follows the Ulam-Hyers stability of the the problem (4.2.7). □

Remark 4.2.4. Of course, we can prove the Ulam-Hyers stability of this problem without this operatorial approach, as in the case of the second proof of the Theorem 4.2.1.

Now we will treat with the stability of Poisson's problem with non homogeneous Neumann boundary conditions.

Theorem 4.2.5. *Let $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega.)$ Then the problem*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g, & \text{on } \partial\Omega \end{cases} \tag{4.2.10}$$

is Ulam-Hyers stable with respect to weak solutions in $H^1(\Omega)/\mathbb{R}$ if and only if

$$\int_\Omega f(x)dx + \int_{\partial\Omega} g(\sigma)d\sigma = 0. \tag{4.2.11}$$

Here we denote by $\int_{\partial\Omega} g(\sigma)d\sigma$ the $d-1$ dimensional surface integral of the function g in the Lebesgue sense.

Proof. We know form Fredholm Alternative theorem, that the compatibility condition (4.2.11) is a necessary and sufficient condition which guarantees the existence of weak solutions in the space $H^1(\Omega)$. Similarly to the previous theorem, we know that we have uniqueness of the solutions in the space $H^1(\Omega)/\mathbb{R}$.

We also know, that there exits a function $u_g \in H^1(\Omega)$, such that $\frac{\partial u_g}{\partial n} = g$ on $\partial\Omega$. So following a similar decomposition procedure as in the case of Theorem 4.2.2 and knowing Theorem 4.2.3 we can deduce the Ulam-Hyers stability of the problem (4.2.10). ☐

Now after these examples we show the Ulam-Hyers stability of general linear elliptic problems.

We want to study the Ulam-Hyers stability with respect to weak solutions of the problem in divergence form

$$\begin{cases} -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + \sum_{i=1}^d b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2.12}$$

The above equation is written for almost every $x \in \Omega$ and the boundary condition also should be satisfied almost everywhere.

It is a well known consequence of the Lax-Milgram theorem, that if $a_{ij}, b_i, c \in L^\infty(\Omega), \forall i, j = 1, \dots d$ and the a_{ij}'s satisfy a coercivity (or an ellipticity) condition, which is

$$\exists \delta > 0 : \sum_{i,j=1}^d a_{ij}(x)\xi^i\xi^j \geq \delta|\xi|^2, \ \forall \xi = (\xi^1, \dots, \xi^d) \in \mathbb{R}^d \text{ and } \forall x \in \Omega, \tag{4.2.13}$$

moreover if we take a constant $\mu \geq 0$ large enough, the modified problem, where in problem (4.2.12) we take $c(x) + \mu$ instead of $c(x)$ has a unique weak solution in $H_0^1(\Omega)$.

For the general case, when we don't deal with a modified problem, but the original one, (4.2.12) by a consequence of the Fredholm Alternative theorem we should have some compatibility condition to be satisfied, in order to guarantee the existence of the weak solutions.

At first we will present a stability result for the modified problem.

Theorem 4.2.6. *Let $f \in H^{-1}(\Omega)$. We assume moreover, that the coefficients satisfy $a_{ij}, b_i, c \in L^\infty(\Omega), \forall i, j = 1, \dots, d$ and the coercivity property* (4.2.13) *holds. Then for a large enough constant $\mu \geq 0$ the problem*

$$\begin{cases} -\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + \sum_{i=1}^d b_i(x)\frac{\partial u}{\partial x_i}(x) + (c(x)+\mu)\,u(x) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2.14}$$

is Ulam-Hyers stable with respect to weak solutions in $H_0^1(\Omega)$.

Proof. We write the weak formulation of the problem (4.2.14). For all $\varphi \in H_0^1(\Omega)$ test function we write

$$B_\mu(u,\varphi) := \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} \varphi + (c+\mu) u \varphi \right) dx$$
$$= < f, \varphi >_{H^{-1} \times H_0^1} .$$

At first we will show the well known result, i.e. the existence and uniqueness of weak solution.

We defined the (not necessarily symmetric) bilinear form B_μ on the space $H_0^1(\Omega) \times H_0^1(\Omega)$.

In order to use the Lax-Milgram theorem, we need the continuity and coercivity of B_μ.

The continuity it can be easily shown, since the coefficient functions are bounded.

We will show the coercivity. This procedure will be useful later on, for the stability result.

Let us take $\varphi = u$, $X := ||\nabla u||_{L^2} = ||u||_{H_0^1}$, $Y := ||u||_{L^2}$ and $M := \max_i ||b_i||_{L^\infty}$.

Then using the property (4.2.13), we can write:

$$B_\mu(u,u) \geq \delta X^2 - MXY + (\mu - ||c_-||_{L^\infty}) Y^2, \tag{4.2.15}$$

where c_- is the negative part of the function c.

We consider

$$q(X,Y) := \delta X^2 - MXY + (\mu - ||c_-||_{L^\infty}) Y^2.$$

If $\mu \geq 0$ is large enough, we have that the discriminant

$$\Delta := M^2 - 4\delta(\mu - ||c_-||_{L^\infty}) < 0,$$

hence there exists an $\alpha > 0 : q(X,Y) \geq \alpha(X^2 + Y^2)$. This implies that $B_\mu(u,u) \geq \alpha ||u||_{H^1}^2 \geq \alpha ||u||_{H_0^1}^2$.

So by the Lax-Milgram theorem we can conclude the existence and uniqueness of the weak solution of the problem (4.2.14).

Now we return to the proof of Ulam-Hyers stability of the problem (4.2.14).

Here we can work either with the operatorial approach or with the standard estimations. We chose the second one.

So we want to study the stability of the following problem:

$$B_\mu(u,\varphi) = < f, \varphi >_{H^{-1} \times H_0^1}, \ \forall \varphi \in H_0^1(\Omega).$$

We want to show, that there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and for all $g \in H^{-1}(\Omega)$, $||f - g||_{H^{-1}} \leq \varepsilon$, there exists a solution of the problem

$$B_\mu(v,\varphi) = < g, \varphi >_{H^{-1} \times H_0^1}, \ \forall \varphi \in H_0^1(\Omega),$$

such that $||u - v||_{H_0^1} \leq C\varepsilon$.

We subtract the two equations and by the bi-linearity and coercivity of the form B_μ and using as test function $\varphi := u - v$, we have

$$\alpha ||u - v||_{H_0^1}^2 \leq B_\mu(u - v, u - v) \leq ||f - g||_{H^{-1}} ||u - v||_{H_0^1}.$$

From here we have

$$||u - v||_{H_0^1} \leq \frac{1}{\alpha}\varepsilon,$$

i.e. the Ulam-Hyers stability of the original problem.

$\square$

Remark 4.2.7. In the case of not a modified problem, i.e. when we do not have $c + \mu$, but only a c function on the left hand side, as we mentioned, if some compatibility condition is satisfied, we will have existence of the weak solutions. In this case we also will have Ulam-Hyers stability, we omit these details.

Remark 4.2.8. We can consider general problems with more general boundary conditions, for example some Robin type conditions, in this case we will also have Ulam-Hyers stability.

4.3 Nonlinear problems

In this section we would like to study the Ulam-Hyers stability of some nonlinear elliptic problems. Let us take an example. We consider the following problem:

$$\begin{cases} -\Delta u(x) = f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.3.1}$$

We remark that the equation is written for a.e. $x \in \Omega$. We study the Ulam-Hyers stability of this problem, assuming some conditions to the function f. We will use some operatorial techniques, because in the case of nonlinear problems this will be more convenient.

At first let us study the case, when f is Lipschitz continuous in the second variable.

Theorem 4.3.1. *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the second variable, with a constant $0 < c < \frac{1}{C_\Omega^2}$ such that*

$$|f(x,t) - f(x,s)| \leq c|t - s|, \forall (x,t), (x,s) \in \Omega \times \mathbb{R}, \tag{4.3.2}$$

where C_Ω is the Poincaré constant, then the problem (4.3.1) *is Ulam-Hyers stable in $H_0^1(\Omega)$.*

Proof. We will study a modified problem at first. Let $\lambda > 0$ a constant (we will give the precise assumptions to it later) and for a $v \in H_0^1(\Omega)$ define

$$\begin{cases} -\Delta u(x) - \lambda\Delta u(x) = -\lambda\Delta v(x) + f(x,v), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.3.3}$$

Let us take an operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ which associates for an input $v \in H_0^1(\Omega)$ the unique solution of the problem (4.3.3). So we transformed our elliptic problem to a fixed point problem. If we can prove that A is a c-weakly PO, then we will be done, because we will have the Ulam-Hyers stability of the fixed point problem, hence of the original elliptic problem. (If we prove that A is a contraction, this would imply, that A is c-weakly PO, and also that the elliptic problem has a unique solution).

Let us take two inputs $v_1, v_2 \in H_0^1(\Omega)$ and let us denote the corresponding solutions $u_i := A(v_i), i = 1, 2$.

Writing the weak formulations and taking the difference, we will have

$$(1+\lambda)\int_\Omega \nabla(u_1 - u_2)\cdot\nabla\varphi = \lambda\int_\Omega \nabla(v_1 - v_2)\cdot\nabla\varphi + \int_\Omega \Big(f(x, v_1) - f(x, v_2)\Big)\varphi,$$

$\forall\varphi \in H_0^1(\Omega)$.

Now we will use some similar steps, as in the case of linear problems.

We choose $\varphi := u_1 - u_2$ and we will have:

$$||A(v_1) - A(v_2)||_{H_0^1}^2 = ||u_1 - u_2||_{H_0^1}^2 \leq \frac{cC_\Omega^2 + \lambda}{1+\lambda}||v_1 - v_2||_{H_0^1}||u_1 - u_2||_{H_0^1}$$

Now dividing the inequality with the positive quantity $||u_1 - u_2||_{H_0^1}$, and because of $c < \frac{1}{C_\Omega^2}$ we will have, that the operator A is a contraction, hence a c-weakly PO.

So from here we will have, that the problems (4.3.3) and (4.3.1) are Ulam-Hyers stable. □

Remark 4.3.2. We will have the same result, if we replace f by an operator $F : H_0^1(\Omega) \to L^2(\Omega)$ with the property that

$$||F(u) - F(v)||_{L^2} \leq c||u - v||_{H_0^1}, \forall u, v \in H_0^1(\Omega),$$

for a given $0 < c < \frac{1}{C_\Omega}$. The only difference is the condition for the constant c, because in this case we have to use just once Poincaré's inequality.

Chapter 5

Further research ideas. Conclusions

In this last chapter we present some further research possibilities and some final remarks and conclusions.

The theory of Picard operators has many-many applications, some of these have been presented in the previous chapters and we recommend also the study of the references. In Mathematics (and of course also in other scientific fields) the obtained results naturally generate always new questions, which could be studied more or less with the same ideas. This is also the case of the presented results from the previous chapters. So we present some of these (open) questions.

5.1 Further research possibilities

We divide these ideas in two little sections.

5.1.1 Stochastic integral inequalities and differential equations

One of the main topics would be the study of stochastic integral inequalities in the framework of Picard operators. The aim would be at first to obtain the upper bounds for stochastic Gronwall type inequalities, using the techniques presented in Chapter 1, i.e. to construct a god notion of metric space (possibly Banach space) for stochastic processes, in order to have a god notion of convergence. Then we can try to adapt some abstract Gronwall inequalities to prove the existing stochastic Gronwall inequalities. Hopefully it would be a god idea to extend the well-known Bielecki type metrics to stochastic processes and try to work in spaces endowed with these type of metrics, norms.

Of course these results would help us to study some more general stochastic integral inequalities, like Wendroff's inequality. If we could adapt the theory of Picard operator to stochastic processes, this fact would also allow us the study of Ulam-Hyers stability of many type of stochastic ODEs and PDEs. By our best knowledge, there have not been obtained yet any results regarding to the Ulam-Hyers stability of stochastic ODEs and PDEs.

We present some known results from [3].

Let $W(t), t \geq 0$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t, t \geq 0$ be the natural filtration of $\mathcal{F}$. For a positive number T, $M_W^2[0,T]$ denotes the set of all separable non-anticipative functions $f(t)$ with respect to $\mathcal{F}_t$ defined on $[0,T]$ satisfying

$$\mathbb{E}\left[\int_0^T f^2(t)dt\right] < +\infty.$$

Theorem 5.1.1 ([3]). *Assume that $\xi(t)$ and $\eta(t)$ belong to $M_W^2[0,T]$. If there exist functions $a(t)$ and $b(t)$ belonging to $M_W^2[0,T]$ such that*

$$|\xi(t)| \leq \left|\int_0^t a(s)ds + \int_0^t b(s)dW(s)\right| \tag{5.1.1}$$

and if there are nonnegative constants $\alpha_0, \alpha_1, \beta_0$ and β_1 such that

$$|a(t)| \leq \alpha_0|\eta(t)| + \alpha_1|\xi(t)|, \quad |b(t)| \leq \beta_0|\eta(t)| + \beta_1|\xi(t)| \tag{5.1.2}$$

for $0 \leq t \leq T$, then we have

$$\mathbb{E}\left(\xi^2(t)\right) \leq 4(\alpha_0\sqrt{t}+\beta_0)^2 \exp\left(4t(\alpha_1\sqrt{t}+\beta_1)^2\right) \int_0^t \mathbb{E}\left(\eta^2(s)\right) ds \tag{5.1.3}$$

for $0 \leq t \leq T$.

So we propose the study of these type of integral inequalities and its generalizations in the framework of Picard operators.

5.1.2 Ulam-Hyers stability of other type of PDEs

As we have seen in Chapter 4, we gave some results for the Ulam-Hyers stability of some linear and non-linear elliptic PDEs in a very general setting, i.e. in Sobolev spaces. Further research ideas would be in this topic to study the stability of more general non-linear problems, where the non-linear part is not Lipschitz continuous, but it has some growth property. Moreover on could study the stability of of problems with more general elliptic operators, which involve for example the $p-$Laplacian operator.

Another objective is to study the Ulam-Hyers stability of evolution equations (including hyperbolic and parabolic problems) in Bochner spaces (time dependent Sobolev spaces). This includes the study of linear and non-linear problems.

For example we can write a general problem, as follows.

$$\begin{cases} \partial_t u + Au = 0, \\ u(0) = u_0 \in E, \end{cases} \tag{5.1.4}$$

where E is a Banach space and $(A, D(A))$ is an $m-$accretive operator. By the famous Hille-Yosida theorem we know that by $m-$accretive property, $(A, D(A))$ generates a C^0 semi-group of contractions (we can denote it by $S(t) := e^{-At}$), with the help of which we can represent the solutions of (5.1.4).

A second objective in the topic of evolution equations would be the study of stability of some nonlinear problems, namely the Ulam-Hyers stability of Hamilton-Jacobi equations with respect to the viscosity solutions. A problem like this can be written as:

$$\begin{cases} \partial_t u + H(t,u,\nabla u) = 0, & \text{in } [0,T]\times\Omega, \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{5.1.5}$$

where the Hamiltonian H has some properties (for example convex).

So basically these are the main topics which could represent the basics of further research in the theory of Ulam-Hyers stability of PDEs.

Of course we could continue our ideas listing many other type of problems, where we could try to use the framework of Picard operator, but this is not our aim right now.

5.2 Final conclusions

In the present thesis we presented two basic type of applications of the theory of Picard operators, developed by Prof. Ioan A. Rus. These are the use of abstract Gronwall lemmas proving (and improving) well-known integral inequalities and secondly the study of the Ulam-Hyers stability of different problems (in different setting).

In Chapter 1 we have obtained the best estimation for the well-known Wendroff's inequality in usual setting, i.e. for some continuous functions defined on closed interval pairs of the real line. This results were motivated mainly by the work of A. Abdeldaim and M. Yakout, who actually tried to give a best estimation to the mentioned inequality in [1], but unfortunately in their paper were more essential errors, hence it was necessary to construct new estimations and proofs. The obtained results were published in [8]. In this paper we also studied some non-linear Wendroff-Bihari type inequalities.

In Chapter 2 we extended the obtained results from Chapter 1 to arbitrary time scales. More precisely we gave improvements to recently obtained results by R. A. C. Ferreira and D. F. M. Torres in [19]. Our proofs are more elegant and powerful in the sense, that we did not use direct methods, but the theory of Picard operators and abstract Gronwall lemmas. The advance of working on arbitrary time scales is that we obtain general results, because time scale calculus unifies the usual continuous and discrete ($q-$)calculus, for example it allows to study differential- and difference equations in the same time. The obtained results from this chapter were published in [9].

In the next chapter we studied the Ulam-Hyers stability of some dynamical equations on arbitrary time scales. We obtained results for bounded time scale intervals and for unbounded ones as well. Here we pointed out that the application of Picard operators to the Ulam-Hyers stability of problems defined on unbounded domains has some disadvantages, it is more difficult to adapt to this case, so we used also some direct methods. Despite this our results are more general than the results of D. R. Anderson in [6] and unifies some results from the theory of differential- and difference equations. The results of this chapter are submitted for publication.

Finally in the next chapter we showed some possible approaches to the Ulam-Hyers stability of some elliptic PDEs in Sobolev spaces through Picard operators. We pointed out here also that in the case of linear elliptic PDEs, the Ulam-Hyers stability does not say some extra information in plus, because it can be obtained just by some simple estimations, using Sobolev embedding theorems, Poincaré's inequality and the Cauchy-Schwartz inequality (we are working on bounded, connected domains with Lipschitz boundary). So in this case there is no need for Picard operators. While in the case of non-linear problems in the present moment we are able to prove only some specific problems, more general problems regarding to Ulam-Hyers stability are still open.

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