1 Express the following complex numbers in the form $x + iy$ with x, y real:
(a) $\frac{(1+2i)(5+2i)}{(2-3i)(1-i)}$, (b) $\frac{1+4i}{1-i} + \frac{1-4i}{1+i}$.

2 Evaluate the product $(1+i)i(1+i)$, first in the "usual algebraic way", then by writing i and $1+i$ in polar form.

3 Write the following complex numbers in polar form:
(a) $2 - i2\sqrt{3}$, (b) $\frac{1}{2+i} - \frac{1}{2-i}$, (c) $\frac{2-i2\sqrt{3}}{1+i}$.

4 Find all complex numbers z for which:
(a) $|\operatorname{Re}(z)| = |z|$, (b) $\operatorname{Im}(z) = |z|$, (c) $|z|^2 = z^2$.

5 If $w = \frac{z-1}{z+1}$ show that $\operatorname{Re}(w) = \frac{|z|^2-1}{|z|^2+2Re(z)+1}$ and $\operatorname{Im}(w) = \frac{2Im(z)}{|z|^2+2Re(z)+1}$.

6 Show that (a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. (b) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

[You could prove these directly by using the definition of conjugation, but there are nicer ways: For (a) note complex conjugation viewed as a reflection in $\mathbb{R}^2$ is a linear map. For (b), you could use the fact that $\overline{z_1 z_2}(z_1 z_2) = |z_1 z_2|^2$.]

7 For each pair of complex numbers z_1 and z_2 prove the parallelogram identity:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\big(|z_1|^2 + |z_2|^2\big).$$

Interpret this equation geometrically.

8 What is the geometric meaning of the following functions $f(z)$ as transformations $z \mapsto f(z)$ from $\mathbb{C}$ to $\mathbb{C}$?
(a) $f(z) = 2z$, (b) $f(z) = -z$, (c) $f(z) = (1+i)z$,
(d) $f(z) = -\bar{z}$, (e) $f(z) = z/|z|$, (f) $f(z) = 1 - i + z$, (g) $f(z) = 1 - i + (1+i)z$.

9 Draw the following sets of points in the complex plane.

$$(a)\ \ z + \bar{z} = 2\ , \qquad (b)\ \ z - \bar{z} = 3i\ , \qquad (c)\ \ |\bar{z}| = 1\ , \qquad (d)\ \ |z - i| = 1\ .$$

10 Solve the following equations in complex numbers, and mark the solutions on a picture of the complex plane:
(a) $|z+2| = |z-2|$, (b) $\bar{z} = 1/z$, (c) $z = \frac{\operatorname{Re} z + \operatorname{Im} z}{2}$, (d) $|(z-2)(\bar{z}-2)| = 1$.

11 What do the following equations represent geometrically? Give sketches.
(i) $|z+2| = 6$ (ii) $|z-3i| = |z+i|$ (iii) $|iz-1| = |iz+1|$ (iv) $|z+1-i| = |\bar{z}-1-i|$.

12 Describe geometrically the subsets of $\mathbb{C}$ specified by
(i) $\operatorname{Im}(z+i) > 2$ (ii) $1 < \operatorname{Re} z \leq 2$ (iii) $|z-1-i| > 1$
(iv) $|z-1+i| \geq |z-1-i|$ (v) $|z+2-i| < |iz-1+2i|$ (vi) $1 < |z-1| < 2$.

13 Show that the equation $|z-a| = \lambda|z-b|$, where a and b are complex numbers and $\lambda > 0$, describes a circle in the complex plane if $\lambda \neq 1$. [In fact, every circle in the complex plane can be written in this form!] What geometric figure is represented when $\lambda = 1$?

14 (i) Apply induction to show De Moivre's formula: $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$.
(ii) Use this to write $\cos(3x)$ as a polynomial in $\cos(x)$; namely show that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$.

15 Write $(1 + i\sqrt{3})^{100}$ in $x + iy$ form.

16 Show that the inverse of the stereographic projection $P : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is given by

$$P^{-1}(z) = \left(\frac{2\,\mathrm{Re}\,(z)}{1+|z|^2}, \frac{2\,\mathrm{Im}\,(z)}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2} \right).$$

17 Consider the inverse stereographic projection $P^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$.
(a) Show that P^{-1} takes the circle $\{z \in \mathbb{C} \mid |z| = c\}$, where $c > 0$ is a given positive number, to a circle on $\mathbb{S}^2 \setminus \{N\}$ that is parallel to the $xy-$plane.
(b)* Explain geometrically why the image of the line $a\,\mathrm{Re}\,(z) + b\,\mathrm{Im}\,(z) = 0$, where $a, b \in \mathbb{R}$ are not both zero, by P^{-1} lies on a great circle on $\mathbb{S}^2 \setminus \{N\}$ that passes via the south pole (in fact – it is the entire circle).

18 Show that $P^{-1}(z) = -P^{-1}(w)$ (i.e. the point $P^{-1}(z)$ and $-P^{-1}(w)$ are diametrically opposite on the Riemann sphere) if and only if $w = -\frac{1}{\bar{z}}$.