Q.1 *Express the following complex numbers in the form $x+iy$ with x, y real:*
(a) $\frac{(1+2i)(5+2i)}{(2-3i)(1-i)}$, *(b)* $\frac{1+4i}{1-i}+\frac{1-4i}{1+i}$.

S.1 (a) $\frac{(1+2i)(5+2i)}{(2-3i)(1-i)}=\frac{1+12i}{-1-5i}=\frac{(1+12i)(-1-5i)}{(-1-5i)(-1+5i)}=\frac{-61-7i}{26}=-\frac{61}{26}-\frac{7}{26}i$.
(b) $\frac{1+4i}{1-i}+\frac{1-4i}{1+i}=\frac{(1+4i)(1+i)+(1-4i)(1-i)}{(1-i)(1+i)}=\frac{(-3+5i)+(-3-5i)}{2}=\frac{-6}{2}=-3$.

Q.2 *Evaluate the product $(1+i)i(1+i)$, first in the "usual algebraic way", then by writing i and $1+i$ in polar form.*

S.2 We have $(1+i)i(1+i)=(i-1)(1+i)=-2$. On the other hand $|1+i|=\sqrt{2}$ so $1+i=\sqrt{2}e^{i\pi/4}$, and $i=e^{i\pi/2}$. Thus, the required product has modulus $\sqrt{2}\,.1\,.\sqrt{2}=2$ and argument $\pi/4+\pi/2+\pi/4=\pi$. The product is therefore $2e^{i\pi}=-2$, as before.

Q.3 *Write the following complex numbers in polar form:*
(a) $2-i2\sqrt{3}$, *(b)* $\frac{1}{2+i}-\frac{1}{2-i}$, *(c)* $\frac{2-i2\sqrt{3}}{1+i}$.

S.3 (a) $|2-i2\sqrt{3}|=\sqrt{4+12}=4$, so we have $2-i2\sqrt{3}=4(\frac{1}{2}-i\frac{\sqrt{3}}{2})=4e^{i5\pi/3}$.
(b) $\frac{1}{2+i}-\frac{1}{2-i}=-\frac{2i}{5}=\frac{2}{5}e^{-i\pi/2}$.
(c) $1+i=\sqrt{2}e^{i\pi/4}$, so we have $\frac{1}{1+i}=\frac{1}{\sqrt{2}}e^{-i\pi/4}$. Thus, $\frac{2-i2\sqrt{3}}{1+i}=4e^{i5\pi/3}\cdot\frac{1}{\sqrt{2}}e^{-i\pi/4}=2\sqrt{2}e^{i17\pi/12}$.

Q.4 *Find all complex numbers z for which:*
(a) $|Re(z)|=|z|$, *(b)* $Im(z)=|z|$, *(c)* $|z|^2=z^2$.

S.4 (a) If $z=x+iy$, then the equation reads $|x|=\sqrt{x^2+y^2}$. This is true if and only if $y=0$ (or equivalently if z is real).
(b) We have $y=\sqrt{x^2+y^2}$. This is true if and only if $x=0$ and $y\geq 0$ (because the real square root function is always non-negative). Equivalently, we must have that z is purely imaginary with $Im(z)\geq 0$.
(c) The equation clearly holds for $z=0$. For $z\neq 0$, we that that $|z|^2=z\bar{z}=z^2$ is equivalent to $z=\bar{z}$, which means z real. So the equation holds if and only if z is real. For example $i^2=-1$, but $|i|=1$.

Q.5 *If $w=\frac{z-1}{z+1}$ show that $Re(w)=\frac{|z|^2-1}{|z|^2+2Re(z)+1}$ and $Im(w)=\frac{2Im(z)}{|z|^2+2Re(z)+1}$.*

S.5 $w=\frac{z-1}{z+1}=\frac{(z-1)(\bar{z}+1)}{(z+1)(\bar{z}+1)}=\frac{|z|^2+z-\bar{z}-1}{|z|^2+z+\bar{z}+1}=\frac{|z|^2-1+2Im(z)i}{|z|^2+2Re(z)+1}$.

Q.6 *Show that* *(a)* $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$. *(b)* $\overline{z_1z_2}=\overline{z_1}\,\overline{z_2}$.

[You could prove these directly by using the definition of conjugation, but there are nicer ways: For (a) note complex conjugation viewed as a reflection in $\mathbb{R}^2$ is a linear map. For (b), you could use the fact that $\overline{z_1z_2}(z_1z_2)=|z_1z_2|^2$.]

S.6 The "nice" ways go as follows: (a) The map $z\mapsto\bar{z}$ when viewed as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ corresponds to $(x,y)\mapsto(x,-y)$. But this map is clearly linear and hence $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$. For (b), we first note the claim holds for if either z_1 or z_2 are zero. Otherwise, we can divide $\overline{z_1z_2}z_1z_2=|z_1z_2|^2=|z_1|^2|z_2|^2=z_1\bar{z}_1z_2\bar{z}_2$ by $z_1z_2\neq 0$ and obtain the claim.

Otherwise: (a) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Then

$$\overline{z_1}+\overline{z_2}=x_1-iy_1+x_2-iy_2=x_1+x_2-i(y_1+y_2),$$

while

$$\overline{z_1+z_2}=\overline{x_1+iy_1+x_2+iy_2}=\overline{x_1+x_2+i(y_1+y_2)}=x_1+x_2-i(y_1+y_2).$$

(b)

$$\overline{z_1}\,\overline{z_2}=(x_1-iy_1)(x_2-iy_2)=x_1x_2-y_1y_2-i(x_1y_2+x_2y_1),$$

while

$$\overline{z_1z_2}=\overline{(x_1+iy_1)(x_2+iy_2)}=\overline{x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)}=x_1x_2-y_1y_2-i(x_1y_2+x_2y_1).$$

Q.7 *For each pair of complex numbers z_1 and z_2 prove the parallelogram identity:*

$$|z_1+z_2|^2+|z_1-z_2|^2=2\big(|z_1|^2+|z_2|^2\big).$$

Interpret this equation geometrically.

S.7

$$\text{LHS}=(z_1+z_2)(\overline{z}_1+\overline{z}_2)+(z_1-z_2)(\overline{z}_1-\overline{z}_2)=2(z_1\overline{z}_1+z_2\overline{z}_2)=\text{RHS}.$$

The equation tells you that the sum of the square of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Q.8 *What is the geometric meaning of the following functions $f(z)$ as transformations $z\mapsto f(z)$ from $\mathbb{C}$ to $\mathbb{C}$?*
(a) $f(z)=2z$, (b) $f(z)=-z$, (c) $f(z)=(1+i)z$,
(d) $f(z)=-\bar{z}$, (e) $f(z)=z/|z|$, (f) $f(z)=1-i+z$, (g) $f(z)=1-i+(1+i)z$.

S.8 (a) Radial expansion away from the origin by a factor of 2.
(b) Reflection in the origin (or, rotation about the origin by π degrees since $-1=e^{i\pi}$).
(c) $(1+i)=\sqrt{2}e^{i\pi/4}$, so rotation about origin through angle $\pi/4$ anticlockwise, followed by a radial expansion away from the origin by a factor of $\sqrt{2}$.
(d) A point $z=x+iy$ goes to $-\bar{z}=-x+iy$, so reflection in the imaginary axis.
(e) Radial projection onto the unit circle.
(f) Translation through $1-i$.
(g) This is (c) followed by (f); i.e., a rotation about origin through angle $\pi/4$ anticlockwise, followed by a radial expansion away from the origin by a factor of $\sqrt{2}$, followed by translation through $1-i$.

Q.9 *Draw the following sets of points in the complex plane.*

$$(a)\ \ z+\bar{z}=2\ , \qquad (b)\ \ z-\bar{z}=3i\ , \qquad (c)\ \ |\bar{z}|=1\ , \qquad (d)\ \ |z-i|=1\ .$$

S.9 (a) Let $z=x+iy$. Then $2=z+\bar{z}=2x$, so the required set is the line $x=1$, parallel to the imaginary axis. (In other words, the set of complex numbers z with $Re(z)=1$.)
(b) Let $z=x+iy$. Then $3i=z-\bar{z}=2yi$, so the required set is those points with imaginary part $y=3/2$; i.e., the line through $3i/2$, parallel to the real axis.
(c) We have $|\bar{z}|=|z|$, and $|z|$ is the distance of z from the origin 0, so $|\bar{z}|=1$ is the circle of radius 1 centre 0.
(d) This is the circle of radius 1 with centre i (so this circle passes through 0 tangential to the real axis).

Q.10 *Solve the following equations in complex numbers, and mark the solutions on a picture of the complex plane:*
(a) $|z+2|=|z-2|$, *(b)* $\bar{z}=1/z$, *(c)* $z=\frac{Re\,z+Im\,z}{2}$, *(d)* $|(z-2)(\bar{z}-2)|=1$.

S.10 The pictures/sketches are not displayed in this solution sheet.
(a) $|z+2|=|z-2|\Leftrightarrow|z+2|^2=|z-2|^2\Leftrightarrow(z+2)(\overline{z}+2)=(z-2)(\overline{z}-2)\Leftrightarrow 2(z+\overline{z})=2(-z-\overline{z})\Leftrightarrow z+\overline{z}=0\Leftrightarrow z$ is purely imaginary. This is clear geometrically because the imaginary axis is the set of point in the complex plane equidistant from 2 and -2.
(b) We must have $z\neq 0$. Then, $\bar{z}=1/z\Leftrightarrow\bar{z}z=1\Leftrightarrow|z|=1$, so $\bar{z}=1/z$ if and only if z lies on the unit circle centre the origin.
(c) Let $z=x+iy$. Then $\operatorname{Re}z+\operatorname{Im}z=x+y$, so that $z=\frac{\operatorname{Re}z+\operatorname{Im}z}{2}$ if and only if $z=0$.
(d) $|(z-2)(\bar{z}-2)|=|(z-2)\overline{(z-2)}|=|z-2||\overline{z-2}|=|z-2|^2$, so the equation simplifies to $|z-2|=1$, and the solutions lie on the circle centred at 2 with radius 1.

Q.11 *What do the following equations represent geometrically? Give sketches.*
(i) $|z+2|=6$ *(ii)* $|z-3i|=|z+i|$ *(iii)* $|iz-1|=|iz+1|$ *(iv)* $|z+1-i|=|\bar{z}-1-i|$.

S.11 (i) $|z+2| = |z-(-2)|$, which is the distance of z from -2, so $|z+2| = 6$ is the circle radius 6 centre -2.
(ii) The equation represents those points with equal distance to $3i$ and $-i$. Thus, the locus is the perpendicular bisector of the line segment joining $3i$ and $-i$; that is, the line parallel to the real axis passing through i. Alternatively, for $z = x+iy$ one could notice $|z-3i| = |z+i| \Leftrightarrow |z-3i|^2 = |z+i|^2 \Leftrightarrow x^2+(y-3)^2 = x^2+(y-1)^2$. Simplifying, we have that this is equivalent to $9-6y = 2y+1$ and in turn $y = 1$.
(iii) $|iz-1| = |iz+1| \iff |i(z+i)| = |i(z-i)| \iff |(z+i)| = |(z-i)|$. This is the perpendicular bisector of the line segment joining $-i$ and i; in other words, the real axis.
(iv) $|z+1-i| = |\bar{z}-1-i| \iff |z-(-1+i)| = |z-(1-i)|$. The perpendicular bisector of $-1+i$ and $1-i$ is the line $y = x$. Alternatively, notice we must have $(x+1)^2+(y-1)^2 = (x-1)^2+(-y-1)^2$, which simplifies to $y = x$.

Q.12 *Describe geometrically the subsets of $\mathbb{C}$ specified by*
(i) Im$(z+i) > 2$ *(ii) $1 <$ Re $z \leq 2$* *(iii) $|z-1-i| > 1$*
(iv) $|z-1+i| \geq |z-1-i|$ *(v) $|z+2-i| < |iz-1+2i|$* *(vi) $1 < |z-1| < 2$.*

S.12 (i) If $z = x+iy$, then $\mathrm{Im}(z+i) > 2 \Leftrightarrow y+1 > 2 \Leftrightarrow y > 1$. Thus, the subset is the area of the plane strictly above the line $y = 1$.
(ii) This is the vertical strip in the (x, y)-plane between the lines $x = 1$ and $x = 2$. The line on the left not included whereas the one on the right is included.
(iii) $|z-1-i| = |z-(1+i)|$ so the equation represents those points z whose distance to $1+i$ is strictly greater than 1. Thus, the equation represents the region outside the circle with centre $1+i$ and radius 1.
(iv) $|z-1+i| = |z-(1-i)|$ and $|z-1-i| = |z-(1+i)|$, so the equation represents the points z closer to $1+i$ than to $1-i$. This is precisely the upper half place (including the real axis).
(v) $|z+2-i| = |z-(-2+i)|$ and $|iz-1+2i| = |i||z-\frac{1}{i}+2| = |z+i+2| = |z-(-2-i)|$. Thus the solutions are those points strictly closer to $-2+i$ than to $-2-i$. This is precisely the upper half plane (with the real axis excluded).
(vi) The equation represents those points whose distance to 1 is between 1 and 2, so the region (called an 'annulus') inside the circle centred at 1 of radius 2 and outside the circle centred at 1 of radius 1.

Q.13 *Show that the equation $|z-a| = \lambda|z-b|$, where a and b are complex numbers and $\lambda > 0$, describes a circle in the complex plane if $\lambda \neq 1$. [In fact, every circle in the complex plane can be written in this form!] What geometric figure is represented when $\lambda = 1$?*

S.13 Put $z = x+iy$ as usual, and let $a = a_1+ia_2$ and $b = b_1+ib_2$. Then $(x-a_1)^2+(y-a_2)^2 = \lambda^2(x-b_1)^2+\lambda^2(y-b_2)^2$. Grouping the terms and factorising we find that this 'simplifies' to

$$\left(x-\frac{a_1-\lambda^2 b_1}{1-\lambda^2}\right)^2+\left(y-\frac{a_2-\lambda^2 b_2}{1-\lambda^2}\right)^2 = \frac{\lambda^2}{1-\lambda^2}\left((a_1-b_1)^2+(a_2-b_2)^2\right).$$

Denoting the terms in the above by $(x-c_1)^2+(y-c_2)^2 = r^2$ we see that $|z-c_1-ic_2| = r$ and the equation represents a circle centred at c_1+ic_2 of radius $r > 0$.

Q.14 *(i) Apply induction to show De Moivre's formula:* $(\cos(x)+i\sin(x))^n = \cos(nx)+i\sin(nx)$.
(ii) Use this to write $\cos(3x)$ *as a polynomial in* $\cos(x)$*; namely show that* $\cos(3x) = 4\cos^3(x)-3\cos(x)$.

S.14 (i) When $n = 1$ the statement is trivial. Assume it holds for $n = k$, then

$$(\cos(x)+i\sin(x))^{k+1} = (\cos(x)+i\sin(x))^k(\cos(x)+i\sin(x)) = (\cos(kx)+i\sin(kx))(\cos(x)+i\sin(x))$$

Expanding gives

$$(\cos(kx)\cos(x)-\sin(kx)\sin(x))+i(\sin(kx)\cos(x)+\sin(x)\cos(kx)) = \cos((k+1)x)+i\sin((k+1)x),$$

by the real double angle formula. Note that in other words we have $\left[e^{ix}\right]^n = (\cos(x)+i\sin(x))^n = \cos(nx)+i\sin(nx) = e^{inx}$.

(ii) Considering the above for $n = 3$, observe that $\cos(3x) = \mathrm{Re}\left((\cos(x) + i\sin(x))^3\right)$. Expanding, we have

$$(\cos(x) + i\sin(x))^3 = (\cos^3(x) - 3\cos(x)\sin^2(x)) + i(3\cos^2(x)\sin(x) - \sin^3(x)).$$

Finally, $\cos(3x) = \cos^3(x) - 3\cos(x)\sin^2(x) = \cos^3(x) - 3\cos(x)(1 - \cos^2(x)) = 4\cos^3(x) - 3\cos(x)$.

Q.15 *Write $(1 + i\sqrt{3})^{100}$ in $x + iy$ form.*

S.15 We have $(1 + i\sqrt{3}) = 2e^{i\pi/3}$, so by De Moivre $(1 + i\sqrt{3})^{100} = 2^{100}e^{100i\pi/3} = 2^{100}e^{99i\pi/3 + i\pi/3} = 2^{100}e^{33i\pi + i\pi/3} = -2^{100}e^{i\pi/3} = -2^{99}(1 + i\sqrt{3})$.

Q.16 *Show that the inverse of the stereographic projection $P : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is given by*

$$P^{-1}(z) = \left(\frac{2\,Re\,(z)}{1 + |z|^2}, \frac{2\,Im\,(z)}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2}\right).$$

S.16 Recall that we have that

$$\mathrm{Re}\,(z) + i\,\mathrm{Im}\,(z) = P(\xi, \eta, \zeta) = \frac{\xi}{1 - \zeta} + i\frac{\eta}{1 - \zeta}$$

and we would like to express $(\xi, \eta, \zeta) \in \mathbb{S}^2 \setminus \{N\}$ with $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$. We have that

$$\mathrm{Re}\,(z) = \frac{\xi}{1 - \zeta}, \qquad \mathrm{Im}\,(z) = \frac{\eta}{1 - \zeta}$$

or

$$\xi = (1 - \zeta)\,\mathrm{Re}\,(z), \qquad \eta = (1 - \zeta)\,\mathrm{Im}\,(z).$$

Since $(\xi, \eta, \zeta) \in \mathbb{S}^2 \setminus \{N\}$ we have that

$$1 = \xi^2 + \eta^2 + \zeta^2 = (1 - \zeta)^2\,\mathrm{Re}\,(z)^2 + (1 - \zeta)^2\,\mathrm{Im}\,(z)^2 + \zeta^2 = (1 - \zeta)^2\,|z|^2 + \zeta^2$$

We can rewrite the above as

$$\left(1 + |z|^2\right)\zeta^2 - 2|z|^2\zeta + |z|^2 - 1 = 0$$

whose solutions are

$$\zeta_{1,2} = \frac{2|z|^2 \pm \sqrt{4|z|^4 - 4\left(1 + |z|^2\right)\left(|z|^2 - 1\right)}}{2\left(1 + |z|^2\right)} = \frac{|z|^2 \pm 1}{1 + |z|^2}.$$

Since $\zeta \neq 1$ we are only left with the option

$$\zeta = \frac{|z|^2 - 1}{1 + |z|^2}.$$

To find ξ and η we notice that

$$1 - \zeta = 1 - \frac{|z|^2 - 1}{1 + |z|^2} = \frac{2}{1 + |z|^2}.$$

Since we found a unique solution for any given z we conclude the desired formula.

Q.17 *Consider the inverse stereographic projection $P^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$.*
(a) Show that P^{-1} takes the circle $\{z \in \mathbb{C} \mid |z| = c\}$, where $c > 0$ is a given positive number, to a circle on $\mathbb{S}^2 \setminus \{N\}$ that is parallel to the $xy-$plane.
(b) Explain geometrically why the image of the line $a\,Re\,(z) + b\,Im\,(z) = 0$, where $a, b \in \mathbb{R}$ are not both zero, by P^{-1} lies on a great circle on $\mathbb{S}^2 \setminus \{N\}$ that passes via the south pole (in fact – it is the entire circle).*

S.17 (a) Assuming that $|z| = c$ we find that

$$P^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{1+c^2}, \frac{2\operatorname{Im}(z)}{1+c^2}, \frac{c^2-1}{1+c^2}\right).$$

The image of the concentric circle is on the plane $z = \frac{c^2-1}{1+c^2}$, i.e. parallel to the $xy-$axis, and since

$$\left(\frac{2\operatorname{Re}(z)}{1+c^2}\right)^2 + \left(\frac{2\operatorname{Im}(z)}{1+c^2}\right)^2 = \frac{4|z|^2}{(1+c^2)^2} = \frac{4c^2}{(1+c^2)^2}$$

we conclude that the image is indeed a circle.
(b) A great circle on $\mathbb{S}^2 \setminus \{N\}$ that passes through the south pole is parametrised by a fixed angle of $\xi\eta-$rotation and a free azimuthal angle. In other words, it is parametrised by

$$\xi(t) = \cos(\theta_0)\sin(t),$$
$$\eta(t) = \sin(\theta_0)\sin(t),$$
$$\zeta(t) = \cos(t),$$

where θ_0 is fixed and is determined by the identity $\tan(\theta_0) = \frac{\eta(t)}{\xi(t)}$ for all $t \in (0, \pi]$. The line $a\operatorname{Re}(z) + b\operatorname{Im}(z) = 0$ satisfies

$$\frac{\eta}{\xi} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} = -\frac{a}{b}$$

when $b \neq 0$ and if $b = 0$ we find that $\operatorname{Re}(z) = 0$ and consequently $\xi = 0$. We conclude that the image of the line is indeed on a great circle that passed through the south pole.

Q.18 *Show that $P^{-1}(z) = -P^{-1}(w)$ (i.e. the point $P^{-1}(z)$ and $-P^{-1}(w)$ are diametrically opposite on the Riemann sphere) if and only if $w = -\frac{1}{\bar{z}}$.*

S.18 We have that $P^{-1}(z) = -P^{-1}(w)$ if and only if

$$\frac{2\operatorname{Re}(z)}{1+|z|^2} = -\frac{2\operatorname{Re}(w)}{1+|w|^2},$$

$$\frac{2\operatorname{Im}(z)}{1+|z|^2} = -\frac{2\operatorname{Im}(w)}{1+|w|^2},$$

$$\frac{|z|^2-1}{1+|z|^2} = \frac{1-|w|^2}{1+|w|^2}.$$

Using the last equation we find that

$$\left(|z|^2-1\right)\left(1+|w|^2\right) = \left(1-|w|^2\right)\left(1+|z|^2\right)$$

or

$$|z|^2 + |z|^2|w|^2 - 1 - |w|^2 = 1 - |w|^2 + |z|^2 - |z|^2|w|^2.$$

The above holds if and only if $|z||w| = 1$. Plugging this into the first equation we find that

$$\operatorname{Re}(w) = -\frac{1+|w|^2}{1+|z|^2}\operatorname{Re}(z) = -\frac{1+\frac{1}{|z|^2}}{1+|z|^2}\operatorname{Re}(z) = -\frac{\operatorname{Re}(z)}{|z|^2} = -\operatorname{Re}\left(\frac{z}{|z|^2}\right) = -\operatorname{Re}\left(\frac{1}{\bar{z}}\right).$$

Similarly $\operatorname{Im}(w) = -\operatorname{Im}\left(\frac{1}{\bar{z}}\right)$. We conclude that $w = -\frac{1}{\bar{z}}$.