1 Show that the ℓ_1-norm and the ℓ_∞-norm do define norms on both $\mathbb{R}^n$ and $\mathbb{C}^n$ (so give rise to metric spaces).

2 Given a finite dimensional real *[or complex]* vector space V with a (positive definite) inner product $\langle\,.\,\rangle$, let

$$d(v,w) := \sqrt{\langle v-w,\ v-w\rangle}$$

for $v, w \in V$. Show directly that d is a metric on V. *[Hint: for property (D3) use Cauchy-Schwarz.]*

3 In the space $C([a,b])$ of continuous functions defined on a closed interval $[a,b]$ (for $a < b$), let

$$d_1(f,g) := \int_a^b |f(t) - g(t)|\, dt.$$

Show that d_1 is a metric on $C([a,b])$.

4 Consider the space $C([a,b])$ of continuous functions on an interval $[a,b]$.

(a) Show that

$$d(f,g) := \max_{x\in[a,b]} |f(x) - g(x)|, \quad f, g \in C([a,b]),$$

defines a metric on $C([a,b])$.

(b) Let $f(x) = x^2 + 7x - 3 \in C([-1,1])$. Describe the open ball $B_2(f)$ and the closed ball $\overline{B}_2(f)$.

5 (i) Let S be any non-empty set. Verify that the standard discrete metric is indeed a metric on S. Hence or otherwise, show that for any $n \in \mathbb{N}$ the function

$$d_n(\boldsymbol{x}, \boldsymbol{y}) := \#\{j : 1 \le j \le n \text{ and } x_j \ne y_j\}, \qquad (\boldsymbol{x} = (x_1,\dots,x_n), \boldsymbol{y} = (y_1,\dots,y_n) \text{ with } x_i, y_i \in S)$$

defines a metric on $S^n := S \times \cdots \times S$ (n times). Here, '$\#A$' denotes the number of elements in a set A. *[When $S = \{0,1\}$ the metric d_2 is the so-called **Hamming metric** in communication theory.]*

(ii) When $S = \mathbb{R}$ and $n = 2$, describe the ball $B_r((0,0))$ in the cases $(a)\, r < 1$; $(b)\, r > 2$; $(c)\, 1 \le r \le 2$.

6 For $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathbb{R}^2$ define the **'Jungle river' metric** on $\mathbb{R}^2$ by

$$d(\boldsymbol{x}, \boldsymbol{y}) := \begin{cases} |x_2 - y_2| & \text{if} \quad x_1 = y_1. \\ |x_2| + |y_2| + |x_1 - y_1| & \text{if} \quad x_1 \ne y_1. \end{cases}$$

(i) Describe geometrically how d measures the distance between two points in $\mathbb{R}^2$ and verify it is a metric.

(ii) Sketch the open balls $B_1(\mathbf{0})$ and $B_4((3,2))$ in $\mathbb{R}^2$ with respect to this metric.

7 (i) Show that in any metric space (X,d) the set $\{x\}$, consisting of a single point $x \in X$, is closed.

(ii) Show that in any metric space (X,d) the closed ball $\overline{B}_r(x) := \{y \in X : d(y,x) \le r\}$, of radius $r > 0$ centred at $x \in X$, is closed.

8 Verify that $\mathbb{H} = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y > 0\}$ and $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \ne 0\}$ and $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ are open subsets of $\mathbb{C}$, but the set $\mathbb{C} \setminus \mathbb{R}_{<0}$ is neither open nor closed in $\mathbb{C}$.

9 Let (X,d) be a metric space. Show that X is "**Hausdorff**"; that is, for any pair of distinct points x and y in X there exist open sets U and V such that x is in U, y is in V, and $U \cap V = \emptyset$. (So in metric spaces we can separate points by open sets.)

10 Let A be a subset of a metric space X. As in lectures, we define the interior A^0 of A by

$$A^0 := \{x \in A : \text{ there exists an open set } U \subseteq A \text{ such that } x \in U\}.$$

(i) Clearly the interior A^0 is open. Show that it is the largest open subset of A; precisely, show that if U is open and $U \subseteq A$ then $U \subseteq A^0$. Deduce that A^0 is the union of all open subsets of A; that is,

$$A^0 = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U.$$

(ii) Show that for any two subsets $A, B \subseteq X$ we have $A^0 \cup B^0 \subseteq (A \cup B)^0$. Write down the interior of the interval $[2, 3)$ in $\mathbb{R}$. Hence or otherwise, give an example of two subsets of $\mathbb{R}$ for which we have $A^0 \cup B^0 \neq (A \cup B)^0$.

11 Let A be a subset of a metric space X. We define the *closure* $\overline{A}$ of A by

$$\overline{A} = \{x \in X : U \cap A \neq \emptyset \text{ for all open sets } U \text{ containing } x\}.$$

(i) Show that $\overline{A} = \{x \in X : \inf_{z \in A} d(x, z) = 0\}$.

(ii) Show directly from the definition that $\overline{A}$ is closed.

12 Consider $\mathbb{R}$ together with the usual metric coming from the absolute value. Show:
(i) The set $\{x\}$, where $x \in \mathbb{R}$ is given, is not open.
(ii) All open intervals are open, all closed intervals are closed.
(iii) Infinite intersections of open sets are not necessarily open.
(iv) The interval $(0, 1]$ is neither open nor closed. What is its closure?

13 Give an example of a metric space X and an $x \in X$ such that $\overline{B}_1(x) \neq \overline{B_1(x)}$; that is, the closure of the open ball is not necessarily the closed ball!!

14 Let A be a subset of a metric space X. Show that We may define the *boundary* ∂A of A by

(i) $\partial A = \{x \in X : \text{for all open sets } U \text{ containing } x, \text{ there exist } y, z \in U \text{ with } y \in A \text{ and } z \in A^c\}$.

(ii) a set A is open if and only if $\partial A \cap A = \emptyset$;

(iii) A is closed if and only if $\partial A \subseteq A$.

15 Show that if a sequence $\{x_n\}$ converges in a discrete metric space, then it is eventually constant.