Q.1 *Show that the ℓ_1-norm and the ℓ_∞-norm do define norms on both $\mathbb{R}^n$ and $\mathbb{C}^n$ (so give rise to metric spaces).*

S.1 Nonnegativity (N1) for both norms is obvious (for either $\mathbb{R}^n$ or $\mathbb{C}^n$). For either $\mathbb{R}^n$ or $\mathbb{C}^n$ we have $\|\lambda \boldsymbol{x}\|_1 = \sum_{i=1}^n |\lambda x_i| = |\lambda| \sum_{i=1}^n |x_i| = |\lambda| \cdot \|\boldsymbol{x}\|_1$, so the ℓ_1-norm satisfies (N2). Also $\|\lambda \boldsymbol{x}\|_\infty = \max_{i=1}^n |\lambda x_i| = |\lambda| \max_{i=1}^n |x_i| = |\lambda| \cdot \|\boldsymbol{x}\|_\infty$ so the ℓ_∞-norm satisfies (N2). Here, $\lambda \in \mathbb{R}$ or $\mathbb{C}$ as required. Finally, for either $\mathbb{R}^n$ or $\mathbb{C}^n$ we have

$$\|\boldsymbol{x}+\boldsymbol{y}\|_1 = \sum_{i=1}^n |x_i + y_i| \le \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1;$$
$$\|\boldsymbol{x}+\boldsymbol{y}\|_\infty = \max_i |x_i + y_i| \le \max_i(|x_i| + |y_i|) \le \max_i |x_i| + \max_i |y_i| = \|\boldsymbol{x}\|_\infty + \|\boldsymbol{y}\|_\infty;$$

since the absolute value/modulus (on $\mathbb{R}$ or $\mathbb{C}$) satisfies the triangle inequality. Thus, both norms satisfy (N3) as required.

Q.2 *Given a finite dimensional real [or complex] vector space V with a (positive definite) inner product $\langle\,.\,\rangle$, let*

$$d(v, w) := \sqrt{\langle v - w,\ v - w \rangle}$$

for $v, w \in V$. Show directly that d is a metric on V. [Hint: for property (D3) use Cauchy-Schwarz.]

S.2 Recall that a (positive definite) inner-product on a real *[or complex]* vector space satisfies the following properties for vectors $u, v, w \in V$:

(I1) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and $\langle \lambda v, w\rangle = \lambda \langle v, w\rangle$ for $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) (linearity in 1st component)

(I2) $\langle v, w\rangle = \langle w, v\rangle$ *[or $\langle v, w\rangle = \overline{\langle w, v\rangle}$ if complex vector space].* (symmetry)

(I3) $\langle v, v\rangle \ge 0$ and $\langle v, v\rangle = 0 \iff v = 0$ (positive definiteness)

Properties (D1) and (D2) for d are obvious (they follow from the (I3) and (I2) respectively). Recall also the *Cauchy-Schwarz* inequality: $|\langle v, w\rangle| \le \sqrt{\langle v, v\rangle} \cdot \sqrt{\langle w, w\rangle}$. Thus, for $u, v, w \in V$ we have $d(v, w)^2$ equals to

$$\begin{aligned}
\langle v - w,\ v - w\rangle &= \langle (v-u) + (u-w),\ (v-u) + (u-w)\rangle \\
&\overset{\text{(I1)}}{=} \langle v - u, v - u\rangle + \langle v - u,\ u - w\rangle + \langle u - w, v - u\rangle + \langle u - w,\ u - w\rangle \\
&\overset{\text{(I2)}}{\le} \langle v - u,\ v - u\rangle + 2|\langle v - u,\ u - w\rangle| + \langle u - w,\ u - w\rangle \\
&\overset{\text{C-S}}{\le} \langle v - u,\ v - u\rangle + 2\sqrt{\langle v - u,\ v - u\rangle} \cdot \sqrt{\langle u - w,\ u - w\rangle} + \langle u - w,\ u - w\rangle \\
&= \left(\sqrt{\langle v - u,\ v - u\rangle} + \sqrt{\langle u - w,\ u - w\rangle}\right)^2 = (d(v, u) + d(u, w))^2
\end{aligned}$$

and so (D3) holds.

Q.3 *In the space $C([a, b])$ of continuous functions defined on a closed interval $[a, b]$ (for $a < b$), let*

$$d_1(f, g) := \int_a^b |f(t) - g(t)|\, dt.$$

Show that d_1 is a metric on $C([a, b])$.

S.3 Note $|f(t) - g(t)| \ge 0$ so that $\int_a^b |f(t) - g(t)|\, dt \ge \int_a^b 0\, dt = 0$. Property (D1) is then obvious from the fact from real analysis that if a continous non-negative function has zero integral over an inverval, then the function is zero on the interval. Property (D2) is obvious. Property (D3) follows from the triangle inequality for $|\,.\,|$ and the linearity of integrals.

Q.4 *Consider the space $C([a, b])$ of continuous functions on an interval $[a, b]$.*

(a) Show that

$$d(f,g) := \max_{x\in[a,b]} |f(x)-g(x)|, \quad f,g \in C([a,b]),$$

defines a metric on $C([a,b])$.

(b) Let $f(x) = x^2 + 7x - 3 \in C([-1,1])$. Describe the open ball $B_2(f)$ and the closed ball $\overline{B}_2(f)$.

S.4 (a) Since $|f(x)-g(x)| \geq 0$ we have that $\max_{x\in[a,b]} |f(x)-g(x)| \geq 0$. Moreover, if $\max_{x\in[a,b]} |f(x)-g(x)| = 0$ we must have that $|f(x)-g(x)| = 0$ for all $x \in [a,b]$ or equivalently, $f = g$. This shows property (D1). Property (D2) is obvious. Property (D3) follows from the fact that for any f, g and h

$$\max_{x\in[a,b]} |f(x)-g(x)| = \max_{x\in[a,b]} |(f(x)-h(x)) + (h(x)-g(x))|$$

$$\leq \max_{x\in[a,b]} (|f(x)-h(x)| + |h(x)-g(x)|) \leq \max_{x\in[a,b]} |f(x)-h(x)| + \max_{x\in[a,b]} +|h(x)-g(x)|.$$

(b)

$$B_2(f) = \{g \in C([a,b]) \mid d(f,g) < 2\} = \left\{g \in C([a,b]) \mid \max_{x\in[a,b]} |g(x)-f(x)| < 2\right\}$$

$$= \{g \in C([a,b]) \mid f(x) - 2 < g(x) < f(x) + 2, \quad \forall x \in [a,b]\}.$$

In other words, $B_2(f)$ is the set of all continuous functions $g \in C([a,b])$ that satisfy

$$x^2 + 7x - 5 < g(x) < x^2 + 7x - 1.$$

Q.5 *(i) Let S be <u>any</u> non-empty set. Verify that the standard discrete metric is indeed a metric on S. Hence or otherwise, show that for any $n \in \mathbb{N}$ the function*

$$d_n(\boldsymbol{x}, \boldsymbol{y}) := \#\{j : 1 \leq j \leq n \text{ and } x_j \neq y_j\}, \qquad (\boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n) \text{ with } x_i, y_i \in S)$$

*defines a metric on $S^n := S \times \cdots \times S$ (n times). Here, '$\#A$' denotes the number of elements in a set A. [When $S = \{0,1\}$ the metric d_2 is the so-called **Hamming metric** in communication theory.]*

(ii) When $S = \mathbb{R}$ and $n = 2$, describe the ball $B_r((0,0))$ in the cases $(a)\, r < 1$; $(b)\, r > 2$; $(c)\, 1 \leq r \leq 2$.

S.5 (i) For the discrete metric $d_0(x,y) := \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y, \end{cases}$ properties (D1) and (D2) are obvious. For (D3), consider $x, y, z \in S$ and consider the two cases:

- <u>$x = y$</u>: From (D1) we trivially have $d(x,z) + d(z,y) \geq 0 = d(x,y)$.
- <u>$x \neq y$</u>: If z coincides with one of x or y, say x, then $d(x,z) + d(z,y) = 0 + 1 = d(x,y)$. If z, x, y are all distinct then $d(x,z) + d(z,y) = 1 + 1 = 2 \geq 1 = d(x,y)$.

For the metric d_n properties (D1) and (D2) are obvious. A nice way of showing property (D3) is to notice that

$$d_n(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} d_0(x_i, y_i), \qquad \text{where } d_0 \text{ is the discrete metric on } S.$$

Since d_0 is a metric we have for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S^n$ that $d_n(\boldsymbol{x}, \boldsymbol{y}) =$

$$\sum_{i=1}^{n} d_0(x_i, y_i) \leq \sum_{i=1}^{n} (d_0(x_i, z_i) + d_0(z_i, y_i)) = \sum_{i=1}^{n} d_0(x_i, z_i) + \sum_{i=1}^{n} d_0(z_i, y_i) = d_n(\boldsymbol{x}, \boldsymbol{z}) + d_n(\boldsymbol{z}, \boldsymbol{y}).$$

(ii) We have $B_r((0,0)) = \{(x_1, x_2) \in \mathbb{R}^2 : d_2((x_1, x_2), (0,0)) < r\}$.
(a) When $r < 1$ we must have $d_2((x_1, x_2), (0,0)) = 0$ and so $B_r((0,0)) = \{(0,0)\}$ is just the origin.
(b) We always have $d_2 \leq 2$, so for $r > 2$ the ball $B_r((0,0)) = \mathbb{R}^2$ is the whole space.
(c) The case $r = 1$ is as in (a). If $1 < r \leq 2$ we must have $d_2((x_1, x_2), (0,0)) = 1$ or 0; so either one component is non-zero or both are. Thus $B_r((0,0))$ is the union of the two axes; $B_r((0,0)) = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$.

Q.6 *For $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathbb{R}^2$ define the **'Jungle river' metric** on $\mathbb{R}^2$ by*

$$d(\boldsymbol{x}, \boldsymbol{y}) := \begin{cases} |x_2 - y_2| & \text{if} \quad x_1 = y_1. \\ |x_2| + |y_2| + |x_1 - y_1| & \text{if} \quad x_1 \neq y_1. \end{cases}$$

(i) Describe geometrically how d measures the distance between two points in $\mathbb{R}^2$ and verify it is a metric.

(ii) Sketch the open balls $B_1(\mathbf{0})$ and $B_4((3,2))$ in $\mathbb{R}^2$ with respect to this metric.

S.6 (i) To get from a point $\boldsymbol{x} = (x_1, x_2)$ to $\boldsymbol{y} = (y_1, y_2)$ we travel from $\boldsymbol{x}$ vertically up or down (as necessary) to the horizontal-axis, then travel along the horizontal axis *[the river!]* the required distance, then travel vertically back up or down to $\boldsymbol{y}$. *[The name comes from the idea that the quickest way to travel through a thick jungle is to climb down the river bank to the river, sail quickly along the river, and then climb the bank again when you get near your destination!]* Properties (D1) and (D2) are obvious from analagous properties of the absolute value. For (D3) we need to split into cases:

$\underline{x_1 = y_1}$: If $z_1 = x_1 = y_1$ then, using properties (N1) and (N3) for the absolute value,

$$d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}) = |x_2 - z_2| + |z_2 + y_2| \overset{(N3)}{\geq} |x_2 - z_2 + z_2 - y_2| = |x_2 - y_2| = d(\boldsymbol{x}, \boldsymbol{y}).$$

If z_1 is distinct from x_1 and y_1, then

$$d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}) = |x_2| + |z_2| + |x_1 - z_1| + |z_2| + |y_2| + |z_1 - y_1| \overset{(N1)}{\geq} |x_2| + |y_2| \overset{(N3)}{\geq} |x_2 - y_2| = d(\boldsymbol{x}, \boldsymbol{y}).$$

$\underline{x_1 \neq y_1}$: If z_1 coincides with one of x_1 and y_1, say $z_1 = x_1$ then

$$d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}) = |x_2 - z_2| + |z_2| + |y_2| + |z_1 - y_1| \overset{(N3)}{\geq} |x_2| + |y_2| + |z_1 - y_1| \overset{(z_1 = x_1)}{=} d(\boldsymbol{x}, \boldsymbol{y}).$$

Finally, if z_1 is distinct from both x_1 and y_1 then

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}) &= |x_2| + |z_2| + |x_1 - z_1| + |z_2| + |y_2| + |z_1 - y_1| \\ &\overset{(N1)}{\geq} |x_2| + |y_2| + |x_1 - z_1| + |z_1 - y_1| \\ &\overset{(N3)}{\geq} |x_2| + |y_2| + |x_1 - y_1| \quad = \quad d(\boldsymbol{x}, \boldsymbol{y}). \end{aligned}$$

(ii) We have $d((x_1, x_2), (0,0)) = \begin{cases} |x_2| & \text{if} \quad x_1 = 0. \\ |x_2| + |x_1| & \text{if} \quad x_1 \neq 0. \end{cases}$ Thus, $B_1((0,0))$ is just the standard unit ball in the ℓ_1-norm - so the interior of a diamond with vertices at $(1,0), (0,1), (-1,0)$ and $(0,-1)$. For $B_4((3,2))$ we have $d((x_1, x_2), (3,2)) = \begin{cases} |x_2 - 2| & \text{if} \quad x_1 = 3. \\ |x_2| + 2 + |x_1 - 3| & \text{if} \quad x_1 \neq 3. \end{cases}$ In the first case ($x_1 = 3$) we have $d < 4$ iff $-2 < x_2 < 6$; this defines a vertical line from the point $(3, -2)$ to the point $(3, 6)$. In the second case ($x_1 \neq 3$) the inequality $d < 4$ defines the ball in the ℓ_1-norm of radius 2 centred at $(3, 0)$; that is, a diamond with vertices at $(5,0), (3,2), (1,0)$ and $(3,-2)$. Thus, the ball $B_4((3,2))$ looks like a diamond with a vertical line coming out of the top (from the tip $(3,2)$ to the point $(3,6)$)!

Q.7 *(i) Show that in any metric space (X, d) the set $\{x\}$, consisting of a single point $x \in X$, is closed.*

(ii) Show that in any metric space (X, d) the closed ball $\overline{B}_r(x) := \{y \in X : d(y, x) \leq r\}$, of radius $r > 0$ centred at $x \in X$, is closed.

S.7 (i) We need to show that the complement of $\{x\}$ is open. Take $y \neq x$. We need $\epsilon > 0$ so that the open ball $B_\epsilon(y)$ does not contain x. Simply let $\epsilon = s = d(x, y) > 0$. Then the ball $B_s(y)$ does not contain x.

(ii) We need to show that complement of $\overline{B}_r(x)$ is open. We use a similar argument to the first part. First, take $y \notin \overline{B}_r(x)$; i.e., take y with $d(x, y) = s > r$. We need an open ball $B_\epsilon(y)$ such that $B_\epsilon(y) \cap \overline{B}_r(x) = \emptyset$. We claim that $B_{s-r}(y)$ works (so take $\epsilon = s - r > 0$). Assume not, then there is an element $z \in B_{s-r}(y) \cap \overline{B}_r(x)$; this means that $d(z, y) < s - r$ and $d(x, z) \leq r$. But then

$$s = d(x, y) \leq d(x, z) + d(z, y) < r + (s - r) = s,$$

which is a contradiction.

Q.8 *Verify that* $\mathbb{H} = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y > 0\}$ *and* $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \neq 0\}$ *and* $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ *are open subsets of* $\mathbb{C}$*, but the set* $\mathbb{C} \setminus \mathbb{R}_{<0}$ *is neither open nor closed in* $\mathbb{C}$.

S.8 In most cases it is straightforward. For $z = x + iy \in \mathbb{H}$ (so $y > 0$) we can consider the ball $B_{y/2}(z)$. Indeed, if $w = a + ib \in B_{y/2}(z)$ then we have that

$$b = \operatorname{Im}(w) = \operatorname{Im}(z) + \operatorname{Im}(w - z) = y + \operatorname{Im}(w - z).$$

Since

$$|\operatorname{Im}(w - z)| \leq |z - w| < \frac{y}{2}$$

we find that

$$b \geq y - |z - w| > y - \frac{y}{2} = \frac{y}{2} > 0.$$

As w was arbitrary we conclude that $B_{y/2}(z) \subset \mathbb{H}$.
For $z \in \mathbb{C}^*$ (so $|z| \neq 0$) we can consider the ball $B_{|z|/2}(z)$. We see that for any $w \in B_{|z|/2}(z$

$$|w| = |z + (w - z)| \geq ||z| - |z - w|| > |z| - \frac{z}{2} = \frac{z}{2} > 0$$

where we have used the reverse triangle inequality

$$|a - b| \geq ||a| - |b||.$$

As w was arbitrary we conclude that $B_{|z|/2}(z) \in \mathbb{C}^*$.
For $z = x + iy \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ a little more care is needed; if $y \neq 0$ (so z is not on the positive real axis) simply pick $B_{y/2}(z) \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (same as with $\mathbb{H}$) and if $y = 0$ (so z is on the positive real axis and $x > 0$) pick $B_{x/2}(z) \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (same ideas as above).

Finally, consider the point $0 \in \mathbb{C} \setminus \mathbb{R}_{<0}$. By definition, any ball $B_\epsilon(0)$ centred at 0 of radius $\epsilon > 0$ must contain the point $z = -\epsilon/2$, since $|(-\epsilon/2) - 0| = \epsilon/2 < \epsilon$, but then $z \notin \mathbb{C} \setminus \mathbb{R}_{<0}$. Thus the set is not open in $\mathbb{C}$. It is not closed in $\mathbb{C}$ because the complement $\mathbb{R}_{<0}$ is clearly not open; to see this simply pick any point $x \in \mathbb{R}_{<0}$ (so $x < 0$) and note that any ball $B_\epsilon(x)$ must contain the point $z = x + i\epsilon/2$, which is not in $\mathbb{R}_{<0}$.

Q.9 *Let* (X, d) *be a metric space. Show that* X *is "**Hausdorff**"; that is, for any pair of distinct points* x *and* y *in* X *there exist open sets* U *and* V *such that* x *is in* U*,* y *is in* V*, and* $U \cap V = \emptyset$*. (So in metric spaces we can separate points by open sets.)*

S.9 Let $x \neq y$ be two points in X and let $r = d(x, y) > 0$ be the distance between them. Then $B_{r/2}(x)$ and $B_{r/2}(y)$ are open (by Lemma 2.6), are disjoint, and containing x and y respectively. To see they are disjoint, assume $z \in B_{r/2}(x) \cap B_{r/2}(y)$; then $d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2 = r$, which is impossible.

Q.10 *Let* A *be a subset of a metric space* X*. As in lectures, we define the interior* A^0 *of* A *by*

$$A^0 := \{x \in A : \textit{ there exists an open set } U \subseteq A \textit{ such that } x \in U\}.$$

(i) Clearly the interior A^0 is open. Show that it is the largest open subset of A; precisely, show that if U is open and $U \subseteq A$ then $U \subseteq A^0$. Deduce that A^0 is the union of all open subsets of A; that is,

$$A^0 = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U.$$

(ii) Show that for any two subsets $A, B \subseteq X$ we have $A^0 \cup B^0 \subseteq (A \cup B)^0$. Write down the interior of the interval $[2, 3)$ in $\mathbb{R}$. Hence or otherwise, give an example of two subsets of $\mathbb{R}$ for which we have $A^0 \cup B^0 \neq (A \cup B)^0$.

S.10 (i) If U is an open subset of A then for every $x \in U$ there exists a ball $B_\epsilon(x) \subseteq U \subseteq A$. But this means $x \in A^0$ and so $U \subseteq A^0$. *[Note, we didn't actually need to use the ball, since U itself is an open set in A containing x - it was good practice however.]*

Since every open subset of A is a subset of A^0, the union of all open subsets (which is open according to a lemma from class) must be a subset of A^0. So,

$$\bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U \subseteq A^0.$$

On the other hand, A^0 is itself an open subset of A. Thus

$$A^0 \subseteq \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U.$$

(ii) Suppose $x \in A^0 \cup B^0$. Then either $x \in A^0$ or $x \in B^0$. In the first case there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$; in the second case there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq B$. In both cases $B_\epsilon(x) \subseteq A \cup B$, which shows that $x \in (A \cup B)^0$ *[Note, we chose a ball $B_\epsilon(x)$ as our open set since if $U \subseteq A$ is open and $x \in U$ we can always find $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U \subseteq A$. We could have used the U given in the definition, though].*

The interior of $(2, 3)$ is clearly $(2, 3)$, since it is open. The interior of $[2, 3)$ is clearly $(2, 3)$; either note that every open ball centred at $x = 2$ (and so every open set containing the point $x = 2$) must contain a point not in $[2, 3)$, or simply note that it is clearly the largest open subset of $[2, 3)$.

For an example of two intervals for which $A^0 \cup B^0 \neq (A \cup B)^0$, you could take $B = [2, 3)$ and $A = (1, 2)$. Then, $(1, 2) \cup (2, 3) = A^0 \cup B^0 \neq (A \cup B)^0 = (1, 3)$.

Q.11 *Let A be a subset of a metric space X. We define the closure $\overline{A}$ of A by*

$$\overline{A} = \{x \in X : U \cap A \neq \emptyset \text{ for all open sets } U \text{ containing } x\}.$$

(i) Show that $\overline{A} = \{x \in X : \inf_{z \in A} d(x, z) = 0\}$.

(ii) Show directly from the definition that $\overline{A}$ is closed.

S.11 (i): For the asserted equality, let $x \in \overline{A}$. Then for all $\epsilon > 0$, there exists a point z in the intersection $B_\epsilon(x) \cap A$ (by definition of $\overline{A}$). So $d(x, z) < \epsilon$. Since $z \in A$, this means $\inf_{z \in A} d(x, z) < \epsilon$. Since ϵ was arbitrary, the infinum must be zero. Conversely, let x be a point in X such that $\inf_{z \in A} d(x, z) = 0$. Consider an open set U containing x. We need to show $U \cap A \neq \emptyset$. Now U contains a small ball $B_\epsilon(x)$. If $B_\epsilon(x) \cap A$ was empty then for all points $z \in A$ we would have $d(x, z) \geq \epsilon$, in contradiction to the assumption that the infimum was zero.

(ii): The simplest way to show this is to use the definition $\overline{A} = \left((A^c)^0\right)^c$ together with the fact that B^0 is an open set for any set B.

Q.12 *Consider $\mathbb{R}$ together with the usual metric coming from the absolute value. Show:*
(i) The set $\{x\}$, where $x \in \mathbb{R}$ is given, is not open.
(ii) All open intervals are open, all closed intervals are closed.
(iii) Infinite intersections of open sets are not necessarily open.
(iv) The interval $(0, 1]$ is neither open nor closed. What is its closure?

S.12 (i) The open ball $B_\epsilon(x)$ in $\mathbb{R}$ is simply the open interval $(x - \epsilon, x + \epsilon)$. Clearly, for every $\epsilon > 0$ this interval is not contained in $\{x\}$, so the set cannot be open.
(ii) Note that we must consider both finite and infinite intervals. We could prove openness "by hand" in every case, but here some "high tech" reasons: Note that an arbitrary finite interval (a, b) is equal to the open ball $B_{(b-a)/2}((a + b)/2)$ and is therefore open by Lemma 2.6. The infinite intervals $(-\infty, b) = \bigcup_n(-n, b)$ and $(a,\infty) = \bigcup_n(a, n)$ are then open since they are the union of open sets (by a result from class).
The complement of the closed interval $[a, b]$ is $(b,\infty) \cup (-\infty, a)$ which is open (since it is the union of two open sets). Thus $[a, b]$ is closed. A similar argument then works for $[b,\infty)$ and $(-\infty, a]$.
(iii) There are many counter-examples. For example, $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. The intervals here are open (by part (ii)), but the set $\{0\}$ is closed.
(iv) For non-openess consider the point 1; any interval of the form $(1-\epsilon, 1+\epsilon)$ must contain the point $1+\epsilon/2$, which is not in $(0, 1]$. For non-closedness, consider the point 0 in the complement $(-\infty, 0] \cup (1,\infty)$; by a similar argument this complement is not open. The closure of $(0, 1]$ is $[0, 1]$ since it is the smallest closed set containing $(0, 1]$.

Q.13 *Give an example of a metric space X and an $x \in X$ such that $\overline{B}_1(x) \neq \overline{B_1(x)}$; that is, the closure of the open ball is not necessarily the closed ball!!*

S.13 There is more than one answer, but one could for example take the subset $X \subseteq \mathbb{R}$ given by $X = \{0, 1\}$, with its usual discrete metric. Let $x = 0$. We know from lectures that all subsets of a discrete metric space are clopen (in particular they are closed) and so $\overline{\{0\}} = \{0\}$. Thus,

$$\overline{B}_1(x) = X = \{0, 1\} \neq \{0\} = \overline{\{0\}} = \overline{B_1(x)}.$$

Q.14 *Let A be a subset of a metric space X. Show that We may define the boundary ∂A of A by*

(i) $\partial A = \{x \in X$: for all open sets U containing x, there exist $y, z \in U$ with $y \in A$ and $z \in A^c\}$.
(ii) a set A is open if and only if $\partial A \cap A = \emptyset$;
(iii) A is closed if and only if $\partial A \subseteq A$.

S.14 (i) Assume that $x \in \partial A = \left(A^0\right)^c \cap \left(\left(A^c\right)^0\right)^c$ and let U be an open set containing x. Since U is open we can find $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. We claim that $B_\epsilon(x) \cap A \neq \emptyset$. Indeed, if the intersection is empty then $B_\epsilon(x) \subseteq A^c$ which shows that x is an interior point of A^c, i.e. $x \in (A^c)^0$. However, as $x \in \partial A$ this contradicts the fact that $x \in \left(\left(A^c\right)^0\right)^c$. We conclude that there exists $y \in B_\epsilon(x) \cap A \subseteq U \cap A$. Replacing A with A^c and using the fact that $\partial A = \partial A^c$ by definition gives us that $B_\epsilon(x) \cap A^c \neq \emptyset$ from which we find $z \in U \cap A^c$.
Conversely, assume that $x \in X$ is such that for any open U such that $x \in U$ we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. We claim that $x \notin A^0$. Indeed, if there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$ then $B_\epsilon(x) \cap A^c = \emptyset$ which is a contradiction to our assumption ($B_\epsilon(x)$ is an open set!). We conclude that $x \in \left(A^0\right)^c$. Replacing A with A^c gives us that $x \in \left(\left(A^c\right)^0\right)^c$ and as such

$$x \in \left(A^0\right)^c \cap \left(\left(A^c\right)^0\right)^c = \partial A.$$

(ii) Assume A is open and let x be a point in A. Then there exists an open ball $B_\epsilon(x)$ in A. But $B_\epsilon(x)$ is an open set containing x that does not contain any point of A^c, so x is not a boundary point. Conversely, if A does not contain any boundary points then for every $x \in A$ there exists an open set U containing x which does not contain any points from A^c; thus $U \subseteq A$. Since U is open there is an open ball $B_\epsilon(x) \subseteq U \subseteq A$ and so A is open.

(iiI) Just note that $\partial A = \partial(A^c)$ and use part (iI): The set A is closed $\iff$ A^c is open $\iff$ $\partial(A^c) \cap (A^c) = \emptyset \iff \partial A \cap (A^c) = \emptyset \iff \partial A \subseteq A$.

Q.15 *Show that if a sequence $\{x_n\}$ converges in a discrete metric space, then it is eventually constant.*

S.15 A sequence $\{x_n\}$ converges to a point x_0 with respect to the discrete metric $d_0(x,y) := \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y, \end{cases}$
if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$ we have $d_0(x_n, x_0) < \epsilon$.

Assuming that $x_n \to x_0$, we notice that for this metric $d_0(x,y) < 1 \iff x = y$. Thus, we may simply take $\epsilon = 1$, say, so that there exists $N \in \mathbb{N}$ with $d_0(x_n, x_0) < 1$ for all $n > N$. Then, for every $n > N$ we have $x_n = x_0$; in other words the sequence is eventually constant.