1 Using the related property for open sets, or otherwise, show that a map $f : X \to Y$ of two metric spaces is continuous if and only if $f^{-1}(F)$ is closed for all closed sets F in Y.

2 Show that a map $f : X \to Y$ between two metric spaces is continuous at $x_0 \in X$ if and only if

$$\lim_{n\to\infty} f(x_n) = f(x_0) \quad \text{for every convergent sequence } \{x_n\}_{n\in\mathbb{N}} \text{ in } X \text{ with } x_n \to x_0.$$

[Hint for '$\Leftarrow$' direction: try a contrapositive argument. In particular, show that if $\exists\, \epsilon > 0$ s.t. $\forall \delta > 0$ $\exists\, x \in B_\delta(x_0)$ with $f(x) \notin B_\epsilon(f(x_0))$, then you can construct a sequence $x_n \to x_0$ with $f(x_n) \not\to f(x_0)$.]

3 Show that $U := \{(x, y) \in \mathbb{R}^2 : \ y - 2x^2 > 0, \ y/x > 2\}$ is open in $\mathbb{R}^2$. *[Hint: Use the properties of continuous functions. But be careful: e.g., y/x is not continuous on $\mathbb{R}^2$ since it is not everywhere defined!]*

4 (i) Show that $U := \{(x, y) \in \mathbb{R}^2 : \ (xy^2 \sin(xy) > 3) \text{ or } (e^{xy-2} + \log(x^2 + 1) < 3y)\}$ is open in $\mathbb{R}^2$.

(ii) Show that $U := \{(x, y) \in \mathbb{R}^2 : \ xy^3/(xy - 1) > 2\}$ is open in $\mathbb{R}^2$.

5 Suppose that X and Y are metric spaces with metrics d_X and d_Y and define a function

$$d : (X \times Y) \times (X \times Y) \to [0, \infty)$$

by $d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$. Check that d defines a metric on $X \times Y$. Show that the map $\pi : X \times Y \to X$ given by $\pi(x, y) = x$ is continuous (with respect to the metrics d and d_X).

6 We call a map $f : (X, d_1) \to (Y, d_2)$ between two metric spaces **Lipschitz** if there exists a positive constant C such that

$$d_2(f(x_1), f(x_2)) \le C d_1(x_1, x_2)$$

for all $x_1, x_2 \in X$. Show that if f is Lipschitz then f is continuous.

7 Let $A \subseteq X$ be a non-empty subset of a metric space X. We define the **distance of a point $x \in X$ to A** by

$$d(x, A) := \inf_{z\in A} d(x, z).$$

Show that the function $f : X \to \mathbb{R} : x \mapsto d(x, A)$ is Lipschitz and hence continuous. *[Hint: Consider, for arbitrary $z \in A$ and $x, y \in X$, the inequality $d(x, z) \le d(x, y) + d(y, z)$ and take the infimum over $z \in A$.]*

8 Consider the map $f : (-\pi, \pi] \to \{z \in \mathbb{C} : |z| = 1\}$ given by $f(t) = e^{it}$. Note that f is a bijection.

(i) Show that f is continuous by using the ϵ-δ definition. (You may use the known facts that $\sin$ and $\cos$ are continuous on $\mathbb{R}$.)

(ii) Prove separately that f is continuous at $t = \pi$ by showing that the preimage $f^{-1}(U_\epsilon)$ is open, where $U_\epsilon = \{e^{it} : t \in (\pi - \epsilon, \pi + \epsilon)\}$ for some $\epsilon > 0$
[Note that due to the fact that f is 2π periodic $U_\epsilon = \{e^{it} : t \in (\pi - \epsilon, \pi] \cup (-\pi, -\pi + \epsilon)\}$].

(iii) However, show that the inverse map $g = f^{-1}$ (the logarithm!) is *not* continuous on $\{z \in \mathbb{C} : |z| = 1\}$ by finding an open set V in $(-\pi, \pi]$ such that $g^{-1}(V)$ is not open. *[Compare this with Q9 below.]*

9 Let $f : X \to Y$ be a bijective continuous map of metric spaces and let X be compact. Show that the inverse map $f^{-1} : Y \to X$ is continuous. *[Hint: Use Q1 and the results from lectures concerning compactness.]*

10 Notice that any complex number can be represented as $z = re^{i\theta}$ for $r \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$. Define a map $f : \mathbb{C} \to \mathbb{R}$ by $f(re^{i\theta}) = r/(\pi + \theta)$, for $r \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$. Show f is not continuous at π, but $f|_L : L \to \mathbb{R}$ is continuous for all straight lines L through the origin.

11 **Discrete Sets**
Let X be any metric space. We call a subset $A \subseteq X$ **discrete** if for every point $x \in A$ there is an open set U containing x that does not intersect any other point of A (in other words, $U \cap A = \{x\}$).

(i) Show that $\mathbb{Z}$ is discrete inside $\mathbb{R}$.
(ii) Show that $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$ is discrete in $\mathbb{R}$, but $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\} \cup \{0\}$ is not.
(iii) Let A be a closed discrete set inside a compact set K. Show that A is finite.
[Hint: assume for a contradiction that A is infinite.]
(iv) Use part (ii) to explain that one needs the "closed" hypothesis in part (iii).
(v) Show that <u>every</u> subset of a discrete metric space is discrete (hence the name).

12 **Connected Sets**
We call a metric space X **connected** if the only subsets which are simultaneously open and closed (that is "clopen") are X and the empty set $\emptyset$. *[Note that this is the opposite situation to a discrete metric space, where <u>every</u> subset is clopen.]*

(i) Let X be the union of the intervals $[0, 1)$ and $[2, 3]$ together with the metric restricted from the standard metric on $\mathbb{R}^n$. Show that X is not connected. *[Note, this explains the terminology.]*
(ii) Show that $\mathbb{R}^n$ is connected (so no proper subset of $\mathbb{R}^n$ is simultaneously open and closed).
[Hint: Assume not, so by definition we could write $\mathbb{R}^n = U \cup V$ with U and V both open and non-empty, say $x \in U, y \in V$. Consider the line segment $\ell(t) := x + t(y - x)$ with $t \in [0, 1]$ from x to y and the "crossing point" from U to V.]

13 **Matrices as metric spaces**
We endow $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, with the norm arising from viewing $M_n(\mathbb{R})$ as $\mathbb{R}^{n^2}$; so $\|A\| = \sqrt{\sum_{i,j} |x_{ij}|^2}$ for any matrix A with entries $x_{ij} \in \mathbb{R}$.

(i) Explain why the determinant is continuous as a map from $M_n(\mathbb{R})$ to $\mathbb{R}$.
(ii) Show that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$.
(iii) Show that $SL_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$, but not compact.
(iv) Recall that $O(n)$, the orthogonal group, consists of column vectors which give a orthonormal basis for $\mathbb{R}^n$. Use Heine-Borel to show that $O(n)$ is compact.
[Hint: To show that $O(n)$ is closed, find (finitely many) continuous functions $f_i : O(n) \to \mathbb{R}$ and closed sets $K_i \subset \mathbb{R}$ such that $O(n) = \bigcap_i f_i^{-1}(K_i)$.]

14 **A different definition of compactness**

(i) Show that x being a limit point of a subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space X is the same thing as $B_r(x)$ containing infinitely many terms x_n of the sequence for any choice of $r > 0$.
(ii) We call a metric space X **compact** if whenever $\{U_i : i \in I\}$ is a collection of open subsets $U_i \subseteq X$ with $X = \bigcup_{i\in I} U_i$, then there exists a *finite* subset $J \subseteq I$ with $X = \bigcup_{i\in J} U_i$. *[We say "Any open cover admits a finite subcover".]* Show that if X is compact then X is compact.
(iii) Suppose X is compact. Show that given $\epsilon > 0$ there exists a finite set of points $x_1, x_2, \ldots, x_r \in X$ such that $X = \bigcup_{i=1}^{r} B_\epsilon(x_i)$. *[This set of points is called a "finite ϵ-net".]*
(iv) Suppose X is compact and $\{U_i : i \in I\}$ is a collection of open subsets $U_i \subseteq X$ with $X = \bigcup_{i\in I} U_i$. Show that there exists $\epsilon > 0$ such that for any point $x \in X$ there exists $i \in I$ with $B_\epsilon(x) \subseteq U_i$.
[Such an ϵ is called a "Lebesgue number" for the cover $\{U_i : i \in I\}$.]
(v) Hence, show that if X is compact then X is compact.

Remark: As mentioned in lectures, our notion of a set being *compact* is more commonly referred to as the set being "sequentially compact". As shown, these two notions coincide for subsets of metric spaces, but they do not in general.