Q.1 *Using the related property for open sets, or otherwise, show that a map $f : X \to Y$ of two metric spaces is continuous if and only if $f^{-1}(F)$ is closed for all closed sets F in Y.*

S.1 If F is closed, then its complement F^c is open. Moreover, every open set $U \subseteq Y$ can be written as $U = (U^c)^c$ i.e. as the complement of a closed set (since U^c is closed by definition). Hence, by a theorem from class, a function f is continuous iff $f^{-1}(F^c)$ is open for every closed set F. However, the set $f^{-1}(F^c) = \big(f^{-1}(F)\big)^c = X \setminus f^{-1}(F)$ being open is equivalent to its complement $f^{-1}(F)$ being closed.

Q.2 *Show that a map $f : X \to Y$ between two metric spaces is continuous at $x_0 \in X$ if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x_0) \quad \textit{for every convergent sequence } \{x_n\}_{n\in\mathbb{N}} \textit{ in } X \textit{ with } x_n \to x_0.$$

[Hint for '$\Leftarrow$' direction: try a contrapositive argument. In particular, show that if $\exists\,\epsilon > 0$ s.t. $\forall\delta > 0$ $\exists\, x \in B_\delta(x_0)$ with $f(x) \notin B_\epsilon(f(x_0))$, then you can construct a sequence $x_n \to x_0$ with $f(x_n) \not\to f(x_0)$.]

S.2 Let d be the metric associated with Y.

- $\Rightarrow$: Let $\epsilon > 0$, assume $x_n \to x_0$, and consider the ball $B_\epsilon(f(x_0))$. By continuity there exists $\delta > 0$ such that if $x \in B_\delta(x_0)$ then $f(x) \in B_\epsilon(f(x_0))$. However, since $x_n \to x_0$ there exists $n \in \mathbb{N}$ such that $x_n \in B_\delta(x_0)$ for any $n > N$. Thus, for every $n > N$ we have $f(x_n) \in B_\epsilon(f(x_0))$; in other words, there exists $n \in \mathbb{N}$ such that for every $n > N$ we have $d(f(x_n), f(x_0)) < \epsilon$, as required.
- $\Leftarrow$: We use a contrapositive argument: we show that if f is not continuous then it does not satisfy the displayed property concerning convergent sequences. So, assume f is not continuous at $x_0 \in X$; that means there exists $\epsilon > 0$ such that for every $\delta > 0$ there is $x \in B_\delta(x_0)$ with $f(x) \notin B_\epsilon(f(x_0))$. *[This is just the negation of the definition of continuity.]* Choose the ϵ for which this holds. Then, in particular, for every $n \in \mathbb{N}$ there exists $x_n \in B_{1/n}(x_0)$ with $f(x_n) \notin B_\epsilon(f(x_0))$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ clearly converges to x, yet for every $n \in \mathbb{N}$ we have $d(f(x_n), f(x_0)) \geq \epsilon$, so $\{f(x_n)\}_{n\in\mathbb{N}}$ cannot converge to $f(x_0)$.

Q.3 *Show that $U := \{(x, y) \in \mathbb{R}^2 : \ y - 2x^2 > 0, \ \ y/x > 2\}$ is open in $\mathbb{R}^2$. [Hint: Use the properties of continuous functions. But be careful: e.g., y/x is not continuous on $\mathbb{R}^2$ since it is not everywhere defined!]*

S.3 Since y/x is not continuous everywhere we cannot use it directly with our theorem relating continuous functions to open sets. Instead notice that

$$U = \{(x, y) \in \mathbb{R}^2 : \ y - 2x^2 > 0, \ \ x > 0, \ \ y - 2x > 0\}$$

Indeed, we have that

$$\big\{(x, y) \in \mathbb{R}^2 : y/x > 2\big\} = \big\{(x, y) \in \mathbb{R}^2 : x > 0, \ y > 2x\big\} \cup \big\{(x, y) \in \mathbb{R}^2 : x < 0, \ y < 2x\big\}$$

and since $y - 2x^2 > 0$ implies $y > 0$ we see that it is impossible to satisfy $y - 2x^2 > 0$ and $x < 0$, $y < 2x$. Letting $f(x, y) = y - 2x^2$, $g(x, y) = x$ and $h(x, y) = y - 2x$ (which are all continous), we have $U = f^{-1}((0,\infty)) \cap g^{-1}((0,\infty)) \cap h^{-1}((0,\infty))$. Since $(0,\infty)$ is open in $\mathbb{R}$, it follows from a theorem from class that all of these preimages are open. U is therefore open since finite intersections of open sets are open.

Q.4 *(i) Show that $U := \{(x, y) \in \mathbb{R}^2 : \ (xy^2 \sin(xy) > 3)$ or $(e^{xy-2} + \log(x^2 + 1) < 3y)\}$ is open in $\mathbb{R}^2$.*

(ii) Show that $U := \{(x, y) \in \mathbb{R}^2 : \ xy^3/(xy - 1) > 2\}$ is open in $\mathbb{R}^2$.

S.4 (i) We have $U = f^{-1}((3,\infty)) \ \cup \ g^{-1}((-\infty, 0))$, where

$$f(x, y) = xy^2 \sin(xy) \quad \text{and} \quad g(x, y) = e^{xy-2} + \log(x^2 + 1) - 3y$$

are continuous functions (they are sums/products/compositions of continuous functions). By Theorem 2.17 each of the above preimages is open, and so U, as a union of open sets, is itself open.

(ii) As in Q4, we must be careful because the function concerned is not continuous on $\mathbb{R}^2$ (since it is not defined when $xy = 1$). To solve this we multiply through by $xy - 1$ and rearrange - but more care is needed; if $xy - 1$ is negative, this act will reverse the inequality. Thus,

$$U = \{(x, y) : xy - 1 > 0,\ xy^3 > 2(xy - 1)\} \cup \{(x, y) : xy - 1 < 0,\ xy^3 < 2(xy - 1)\}$$

Hence

$$U = \left(f^{-1}((0, \infty)) \cap g^{-1}((0, \infty))\right) \cup \left(f^{-1}((-\infty, 0)) \cap g^{-1}((-\infty, 0))\right)$$

where $f(x, y) = xy - 1$ and $g(x, y) = xy^3 - 2(xy - 1)$ are continuous. This is open since finite intersections and arbitrary unions of open sets are open.

Q.5 *Suppose that X and Y are metric spaces with metrics d_X and d_Y and define a function*

$$d : (X \times Y) \times (X \times Y) \to [0, \infty)$$

by $d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$. Check that d defines a metric on $X \times Y$. Show that the map $\pi : X \times Y \to X$ given by $\pi(x, y) = x$ is continuous (with respect to the metrics d and d_X).

S.5 It is clear that $d((x, y), (x', y'))$ is non-negative and only 0 when $x = x'$ and $y = y'$; that is, when $(x, y) = (x', y')$. So, property (D1) holds. Also the symmetry property of d follows from the symmetry properties (D2) of d_X and d_Y. All that remains is to check the triangle inequality (D3). Let $x, x', x'' \in X$ and $y, y', y'' \in Y$. Then

$$\begin{aligned} d((x, y), (x', y')) &= d_X(x, x') + d_Y(y, y') \\ &\leq d_X(x, x'') + d_X(x'', x') + d_Y(y, y'') + d_Y(y'', y') \\ &= d((x, y), (x'', y'')) + d((x'', y''), (x', y')). \end{aligned}$$

We now prove continuity. Let $(x, y) \in X \times Y$ and let $\epsilon > 0$. Then for every $(x', y') \in X \times Y$ such that $d((x, y), (x', y')) < \epsilon$ we have

$$\epsilon > d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y') \overset{(D1)}{\geq} d_X(x, x') = d_X(\pi((x, y)), \pi((x', y'))).$$

Hence, taking $\delta = \epsilon$ is sufficient to establish continuity at (x, y).

Q.6 *We call a map $f : (X, d_1) \to (Y, d_2)$ between two metric spaces* ***Lipschitz*** *if there exists a positive constant C such that*

$$d_2(f(x_1), f(x_2)) \leq C d_1(x_1, x_2)$$

for all $x_1, x_2 \in X$. Show that if f is Lipschitz then f is continuous.

S.6 Let $\epsilon > 0$. We will show f is continous at $x_1 \in X$. To do this we need to find $\delta > 0$ such that for every $x_2 \in X$ with $d_1(x_1, x_2) < \delta$ we have $d_2(f(x_1), f(x_2)) < \epsilon$.

We may simply take $\delta = \epsilon/C$. Indeed, by the definition of Lipschitz, for any $x_2 \in X$ with $d_1(x_1, x_2) < \delta = \epsilon/C$ we have

$$d_2(f(x_1), f(x_2)) \leq C d_1(x_1, x_2) < C\epsilon/C = \epsilon,$$

and so f is continous.

Q.7 *Let $A \subseteq X$ be a non-empty subset of a metric space X. We define the* ***distance of a point $x \in X$ to A*** *by*

$$d(x, A) := \inf_{z \in A} d(x, z).$$

Show that the function $f : X \to \mathbb{R} : x \mapsto d(x, A)$ is Lipschitz and hence continuous. [Hint: Consider, for arbitrary $z \in A$ and $x, y \in X$, the inequality $d(x, z) \leq d(x, y) + d(y, z)$ and take the infimum over $z \in A$.]

S.7 We will show $|f(x) - f(y)| \leq d(x, y)$ and so Lipschitz with constant $C = 1$. For arbitrary $z \in A$ and $x, y \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$ by property (D3). Now take the infimum over $z \in A$, and we obtain

$$f(x) \leq d(x, y) + f(y).$$

Thus $f(x) - f(y) \leq d(x, y)$. By swapping the roles of x and y and repeating the procedure we also have $f(y) - f(x) \leq d(x, y)$, and the claim follows.

Q.8 *Consider the map $f : (-\pi, \pi] \to \{z \in \mathbb{C} : |z| = 1\}$ given by $f(t) = e^{it}$. Note that f is a bijection.*

(i) Show that f is continuous by using the ϵ-δ definition. (You may use the known facts that $\sin$ *and* $\cos$ *are continuous on $\mathbb{R}$.)*

(ii) Prove separately that f is continuous at $t = \pi$ by showing that the preimage $f^{-1}(U_\epsilon)$ is open, where $U_\epsilon = \{e^{it} : t \in (\pi - \epsilon, \pi + \epsilon)\}$ for some $\epsilon > 0$
[Note that due to the fact that f is 2π periodic $U_\epsilon = \{e^{it} : t \in (\pi - \epsilon, \pi] \cup (-\pi, -\pi + \epsilon)\}$].

(iii) However, show that the inverse map $g = f^{-1}$ (the logarithm!) is not continuous on $\{z \in \mathbb{C} : |z| = 1\}$ by finding an open set V in $(-\pi, \pi]$ such that $g^{-1}(V)$ is not open. [Compare this with Q9 below.]

S.8 (i) Fix $t_0 \in (-\pi, \pi]$ and let $\epsilon > 0$. To show f is continous at t_0 we need to find $\delta > 0$ such that for every $t \in (-\pi, \pi]$ satisfying $|t - t_0| < \delta$ in $\mathbb{R}$ we have $|f(t) - f(t_0)| < \epsilon$ in $\mathbb{C}$. Well,

$$|f(t) - f(t_0)| = |e^{it} - e^{it_0}| = |(\cos(t) - \cos(t_0)) + i(\sin(t) - \sin(t_0))|.$$

Now, note that both $\cos$ and $\sin$ are continuous functions on $(-\pi, \pi]$ so, in particular, there exist $\delta_c > 0$ and $\delta_s > 0$ such that for all $t \in (-\pi, \pi]$

$$|t - t_0| < \delta_c \Rightarrow |\cos(t) - \cos(t_0)| < \epsilon/\sqrt{2} \quad \text{and} \quad |t - t_0| < \delta_s \Rightarrow |\sin(t) - \sin(t_0)| < \epsilon/\sqrt{2}.$$

Now take $\delta = \min(\delta_c, \delta_s)$. Then, for every $t \in (-\pi, \pi]$ with $|t - t_0| < \delta$ we have

$$|f(t) - f(t_0)| = \sqrt{(\cos(t) - \cos(t_0))^2 + (\sin(t) - \sin(t_0))^2} < \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon,$$

as required.

(ii) First notice that as subsets of the metric space $(-\pi, \pi]$ the intervals $(\pi - \epsilon, \pi]$ and $(-\pi, -\pi + \epsilon)$ are both open in $(-\pi, \pi]$ with respect to the restricted metric. Following the remark $f^{-1}(U_\epsilon) = (\pi - \epsilon, \pi] \cup (-\pi, -\pi + \epsilon)$, which, as the union of two open sets, must also be open.

(iii) For example, set $V_\epsilon := (\pi - \epsilon, \pi]$, which is open in $(-\pi, \pi]$. Then $g^{-1}(V_\epsilon) = f(V_\epsilon) = \{e^{it} : t \in (\pi - \epsilon, \pi]\}$, which is not open in $\{z \in \mathbb{C} : |z| = 1\}$.

Q.9 *Let $f : X \to Y$ be a bijective continuous map of metric spaces and let X be compact. Show that the inverse map $f^{-1} : Y \to X$ is continuous. [Hint: Use Q1 and the results from lectures concerning compactness.]*

S.9 We start by noticing that for invertible function the pre-image of a set U is the image of the inverse function of the same set (by definition).
By Q1, we need to show that $(f^{-1})^{-1}(F) = f(F)$ is closed in Y for every closed set F in X. From class we know that any closed subset of a compact set is compact, so every such F must also be compact. By another theorem from class we know that the image of any compact set under a continuous map is compact. Thus, $f(F)$ is compact; in particular $f(F)$ is indeed closed.

Q.10 *Notice that any complex number can be represented as $z = re^{i\theta}$ for $r \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$. Define a map $f : \mathbb{C} \to \mathbb{R}$ by $f(re^{i\theta}) = r/(\pi + \theta)$, for $r \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$. Show f is not continuous at π, but $f|_L : L \to \mathbb{R}$ is continuous for all straight lines L through the origin.*

S.10 We will solve this problem by using the sequential criterion of continuity (see Q2). We start by noticing that any line L passing through the origin is the union the origin with two rays

$$L_{1,\theta_0} = \left\{ z \in \mathbb{C} \ : \ z = re^{i\theta_0}, \ r > 0 \right\}, \qquad L_{2,\theta} = \left\{ z \in \mathbb{C} \ : \ z = re^{i(\pi-\theta_0)}, \ r > 0 \right\},$$

where θ_0 is a fixed angle. This means that any sequence $\{z_n\}_{n\in\mathbb{N}}$ on L is represented by a sequence $\{r_n\}_{n\in\mathbb{N}}$ in $R_{\geq 0}$ and a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ which can attain two possible values: θ_0 or $\pi - \theta_0$. Since $r_n = |z_n|$ we see that if $\{z_n\}_{n\in\mathbb{N}}$ converges to z then $\{r_n\}_{n\in\mathbb{N}}$ must converge to $|z|$. Next we notice that if $z \neq 0$ then since $B_{\frac{|z|}{2}}(z)$ must contain all but finitely many $\{z_n\}$ –s and is separated from the origin, θ_n must be either θ_0 or $\pi - \theta_0$ for all but finitely many $\{z_n\}$ –s. Without loss of generality, let us assume that it is θ_0, and notice that this is exactly Arg (z). We conclude that for all but finitely many n–s

$$f(z_n) = \frac{r_n}{\pi + \theta_n} = \frac{r_n}{\pi + \theta_0} \underset{n\to\infty}{\longrightarrow} \frac{|z|}{\pi + \theta_0} = f\left(|z|e^{i\theta_0}\right) = f(z).$$

When $z = 0$ we find that

$$\left|\frac{1}{\pi + \theta_n}\right| \leq \max\left\{\frac{1}{\pi + \theta_0}, \frac{1}{2\pi - \theta_0}\right\}$$

and as such

$$0 \leq |f(z_n)| \leq \max\left\{\frac{1}{\pi + \theta_0}, \frac{1}{2\pi - \theta_0}\right\} r_n$$

As the right hand side goes to zero we can use the pinching lemma to conclude that

$$\lim_{n\to\infty} f(z_n) = 0 = f(0)$$

and conclude the continuity on lines.

To show the lack of continuity at π we consider the sequence $z_n = e^{i\left(-\pi + \frac{1}{n}\right)}$. We have that $z_n \underset{n\to\infty}{\longrightarrow} e^{-i\pi} = e^{i\pi}$ and as such

$$f\left(\lim_{n\to\infty} z_n\right) = f\left(e^{i\pi}\right) = \frac{1}{2\pi}.$$

On the other hand

$$f(z_n) = \frac{1}{\pi + \left(-\pi + \frac{1}{n}\right)} = n \underset{n\to\infty}{\longrightarrow} \infty.$$

Q.11 **Discrete Sets**

*Let X be any metric space. We call a subset $A \subseteq X$ **discrete** if for every point $x \in A$ there is an open set U containing x that does not intersect any other point of A (in other words, $U \cap A = \{x\}$).*

(i) Show that $\mathbb{Z}$ is discrete inside $\mathbb{R}$.

(ii) Show that $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$ is discrete in $\mathbb{R}$, but $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\} \cup \{0\}$ is not.

(iii) Let A be a closed discrete set inside a compact set K. Show that A is finite. [Hint: assume for a contradiction that A is infinite.]

(iv) Use part (ii) to explain that one needs the "closed" hypothesis in part (iii).

(v) Show that every subset of a discrete metric space is discrete (hence the name).

S.11 (i) For each $n \in \mathbb{Z}$, simply consider the ball $B_{1/2}(n)$ of radius $1/2$ around n. This does not intersect any other integers.

(ii) We must be a little careful. When $n = \pm 1$ the ball $B_{1/2}(1/n)$ does not intersect any other point in the set $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$. For $n \neq \pm 1$ the difference between any two consecutive members of the set is

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)},$$

and similarly, $\frac{1}{n-1}-\frac{1}{n}=\frac{1}{n(n-1)}$. Thus, we may take, for any fixed $n \in \mathbb{Z}\backslash\{0\}$, $\epsilon_n = \min\left(\frac{1}{n(n+1)}, \frac{1}{n(n-1)}\right) > 0$; which of the two quantities is the smallest depends on whether n is positive or negative. Then $B_{\epsilon_n}(1/n)$ does not intersect any other point in the set.

On the other hand, for the second set any ball $B_r(0)$ around 0 will intersect $\{\frac{1}{n}\}$ for some n, so the set is not discrete.

(iii) Assume A was infinite and let $\{x_n\}$ be an infinite sequence of *different* points in A. Since A is closed and inside a compact set, it is itself compact (by a result from class). Hence there exists a convergent subsequence with limit $x \in A$ (which is different from all but possibly one element in the sequence). But this is impossible, since this means that any ball around x must contain infinitely many elements of the sequence, contradicting the discreteness of A.

(iv) The set $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$ is discrete in the compact interval $[-1, 1]$, but is infinite. On the other hand, $\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$ is not closed; there is no open ball centred at the origin and entirely contained in the complement of the set.

(v) Let Y be a subset of a discrete metric space X with metric $d(x,y) := \begin{cases} 0 & \text{if} \quad x = y. \\ 1 & \text{if} \quad x \neq y. \end{cases}$ Simply take $\epsilon = 1/2$. Then for every point $y \in Y$ we have $B_\epsilon(y) = \{y\}$, and (since open balls are open sets) Y is discrete. *[Note, the empty set $\emptyset$ is trivially discrete.]*

Q.12 **Connected Sets**

*We call a metric space X **connected** if the only subsets which are simultaneously open and closed (that is "clopen") are X and the empty set $\emptyset$. [Note that this is the opposite situation to a discrete metric space, where every subset is clopen.]*

(i) *Let X be the union of the intervals $[0, 1)$ and $[2, 3]$ together with the metric restricted from the standard metric on $\mathbb{R}^n$. Show that X is not connected. [Note, this explains the terminology.]*

(ii) *Show that $\mathbb{R}^n$ is connected (so no proper subset of $\mathbb{R}^n$ is simultaneously open and closed).*
[Hint: Assume not, so by definition we could write $\mathbb{R}^n = U \cup V$ with U and V both open and non-empty, say $x \in U, y \in V$. Consider the line segment $\ell(t) := x + t(y - x)$ with $t \in [0, 1]$ from x to y and the "crossing point" from U to V.]

S.12 (i) We have that in X the open ball $B_1(1/2) = \{x \in X : |x - 1/2| < 1\}$ of radius 1 around $1/2$ is equal to $[0, 1)$! Hence $[0, 1)$ is an open set in X (but of course not in $\mathbb{R}$). Similarly $B_1(5/2) = [2, 3]$, so the latter interval is also open in X! Since each of $[0, 1)$ and $[2, 3]$ are complements of each other in X, they are also both closed. Thus, we have two clopen subsets not equal to X or $\emptyset$.

(ii) For a contradiction, assume that U is a proper clopen subset of $\mathbb{R}^n$ (so that U is not $\mathbb{R}^n$ or $\emptyset$). Then its complement $V := \mathbb{R}^n \setminus U$ is also clopen (since it is the complement of an open set and the complement of a closed set). In particular we have that $\mathbb{R}^n = U \cup V$ is the disjoint union of two non-empty open sets.

With the notation from the hint, we let $s := \sup(t \in [0, 1] : \ell(t) \in U)$ determine the "crossing point" $z := \ell(s)$, where the line leaves U and enters V. Since $\ell(0) = x \in U$, this number is well defined. If $z \in U$, then there must exists a ball $B_\epsilon(z)$ around z that lies inside U, since U is open. But this is impossible, since by construction $\ell(t)$ lies in V for $t \in (s, s + \epsilon)$. If $z \in V$ we argue similarly; there must exists a ball $B_\epsilon(z)$ around z lying inside V, since V is open, which is also impossible since $\ell(t)$ for $t \in (s - \epsilon, s)$ lies in U.

[Note, essentially we prove "path-connected" implies "connected".]

Q.13 **Matrices as metric spaces**

We endow $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, with the norm arising from viewing $M_n(\mathbb{R})$ as $\mathbb{R}^{n^2}$; so $\|A\| = \sqrt{\sum_{i,j} |x_{ij}|^2}$ for any matrix A with entries $x_{ij} \in \mathbb{R}$.

(i) *Explain why the determinant is continuous as a map from $M_n(\mathbb{R})$ to $\mathbb{R}$.*

(ii) Show that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$.

(iii) Show that $SL_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$, but not compact.

(iv) Recall that $O(n)$, the orthogonal group, consists of column vectors which give a orthonormal basis for $\mathbb{R}^n$. Use Heine-Borel to show that $O(n)$ is compact.
[Hint: To show that $O(n)$ is closed, find (finitely many) continuous functions $f_i : O(n) \to \mathbb{R}$ and closed sets $K_i \subset \mathbb{R}$ such that $O(n) = \bigcap_i f_i^{-1}(K_i)$.]

S.13 (i) Notice that the determinant is a polynomial map in the entries of the matrix. *[To be precise,*

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(i),i} \qquad \textit{where the sum is over all permutations } \sigma.]$$

Thus the determinant is continuous. *[Alternatively, note that for a 1×1 matrix the determinant is just the identity map and is therefore trivially continuous. Then the determinant of an $n \times n$ matrix is just a sum/product of projection maps and the determinants of the minors (which are continuous by induction).]*

(ii) $GL_n(\mathbb{R})$ is the preimage of the union of open sets $(-\infty, 0) \cup (0, \infty)$ in $\mathbb{R}$ under the determinant. Since this union of open sets is an open set, the set $GL_n(\mathbb{R})$ is open.

(iii) $SL_n(\mathbb{R})$ is the preimage of $\{1\} \in \mathbb{R}$ under the determinant and since $\{1\}$ is closed in $\mathbb{R}$ and the determinant is continuous we conclude that $SL_n(\mathbb{R})$ is closed. To be compact, $SL_n(\mathbb{R})$ also needs to be bounded. But, for example, the diagonal matrices of the form $\operatorname{diag}(t, t^{-1}, 1, \cdots, 1)$ with $t \in \mathbb{R}_{\neq 0}$ are contained in $SL_n(\mathbb{R})$, but are not bounded; for any potential bound R we can find t such that

$$\|\operatorname{diag}(t, t^{-1}, 1, \cdots, 1)\| = \sqrt{t^2 + (1/t)^2 + (n-2)(1)^2}$$

is greater than R.

(iv) We show the set is closed and bounded (where we use the view of $M_n(\mathbb{R})$ as $\mathbb{R}^{n^2}$ and apply Heine-Borel). An orthonormal basis consists of (pairwise orthogonal) vectors $\{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$ with $\|\boldsymbol{x_i}\|_n = 1$. Hence, the norm of any matrix $X = (\boldsymbol{x_1}, \cdots, \boldsymbol{x_n})$ in $O(n)$ is given by $\|X\| = \sqrt{n}$. Thus, $O(n)$ is bounded.

For closedness, we want to write $O(n)$ as the preimage of a closed set under a continuous function - there is no obvious way to write $O(n)$ as a preimage under the determinant, so lets think of another continuous map. Define the maps $f_{ij} : M_n(\mathbb{R}) \to \mathbb{R}$ on the set $M_n(\mathbb{R})$ by $f_{ij}(X) = \boldsymbol{x_i}^T \cdot \boldsymbol{x_j}$, where $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$ denote the i-th and j-th column vectors of X respectively. Note that these finitely many maps are all continuous (since they are just polynomials in the matrix entries). Then

$$O(n) = \bigcap_{i \neq j} f_{ij}^{-1}(\{0\}) \cap \bigcap_{i} f_{ii}^{-1}(\{1\}).$$

Each of the preimages $f_{ij}^{-1}(\{0\})$ and $f_{ii}^{-1}(\{1\})$ is closed by Theorem 2.17. Thus, the set $O(n)$ is the finite intersection of closed sets, thus closed.

Q.14 **A different definition of compactness**

(i) Show that x being a limit point of a subsequence of a sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space X is the same thing as $B_r(x)$ containing infinitely many terms x_n of the sequence for any choice of $r > 0$.

(ii) We call a metric space X ***compact*** *if whenever $\{U_i : i \in I\}$ is a collection of open subsets $U_i \subseteq X$ with $X = \bigcup_{i \in I} U_i$, then there exists a* finite *subset $J \subseteq I$ with $X = \bigcup_{i \in J} U_i$. [We say "Any open cover admits a finite subcover".] Show that if X is compact then X is compact.*

(iii) Suppose X is compact. Show that given $\epsilon > 0$ there exists a finite set of points $x_1, x_2, \ldots, x_r \in X$ such that $X = \bigcup_{i=1}^{r} B_\epsilon(x_i)$. [This set of points is called a "finite ϵ-net".]

(iv) Suppose X is compact and $\{U_i : i \in I\}$ is a collection of open subsets $U_i \subseteq X$ with $X = \bigcup_{i \in I} U_i$. Show that there exists $\epsilon > 0$ such that for any point $x \in X$ there exists $i \in I$ with $B_\epsilon(x) \subseteq U_i$. [Such an ϵ is called a "Lebesgue number" for the cover $\{U_i : i \in I\}$.]

(v) Hence, show that if X is compact then X is compact.

Remark: *As mentioned in lectures, our notion of a set being compact is more commonly referred to as the set being "sequentially compact". As shown, these two notions coincide for subsets of metric spaces, but they do not in general.*

S.14 (i) If x_{n_i} is a subsequence with a limit point x, and let $r > 0$. Then by the definition of convergence, all points x_{n_i} are in $B_r(x)$ for all large enough values of i. Conversely, suppose $B_r(x)$ contains infinitely many terms of the sequence x_n for any choice of r. Let x_{n_1} be a member of the sequence in $B_1(x)$. Then, for $i > 1$, let x_{n_i} be any member of the sequence in $B_{1/i}(x)$ with $n_i > n_{i-1}$. It is then straightforward to see that x_{n_i} converges to x.

(ii) Suppose X is compact, and let x_n be a sequence of points in X. Suppose for a contradiction that there is no convergent subsequence of x_n with limit point in X. We now use part (i). For each $x \in X$ we may choose $r_x > 0$ such that $B_{r_x}(x)$ contains only finitely many terms of this sequence (because otherwise x_n would be a convergent subsequence with limit x). Then, of course, we have $X = \bigcup_{x \in X} B_{r_x}(x)$. Since X is compact only finitely many of the balls are needed to cover X; that is, there exist a finite number of points $y_i \in X$ (for $i = 1, 2, \ldots, k$, say) such that $X = \bigcup_{i=1}^{k} B_{r_{y_i}}(y_i)$. But each of these balls contains only finitely many members of terms of the sequence. This means the sequence can only have finitely many terms - a contradiction. *[Note, for this to be a contradiction we actually need to assume initially that x_n contained infinitely many distinct points from X, but if this was not the case there would be a point y repeated infinitely often. There is trivially a convergent subsequence in this case, namely $\{y, y, y, y, \ldots\}$.]*

(iii) We prove this via contrapositive. Let $x_1 \in X$ and assume the covering property fails for some $\epsilon_0 > 0$. Then, we can define the infinite sequence $x_1, x_2, x_3, \ldots$ inductively by choosing x_{n+1} to be in the complement of $\bigcup_{i=1}^{n} B_{\epsilon_0}(x_i)$ (since $\{x_1, x_2, \ldots, x_n\}$ is not a finite ϵ_0-net then there exists a point in the complement). But then it is clear that the sequence $\{x_i\}$ has no convergent subsequence since $d(x_i, x_j) \geq \epsilon_0$ for all i, j.

(iv) Suppose by contradiction that X is compact and that there does not exist such an $\epsilon > 0$. Then for all positive integers n there exists $x_n \in X$ with $B_{1/n}(x_n)$ not contained in any of the open sets U_i. Let x_{n_m} be a convergent subsequence and call the limit x. Now $x \in U_i$ for at least one $i \in I$ since the U_i cover X. Also, U_i is open so there exists $\delta > 0$ such that $B_\delta(x) \subseteq U_i$. Then choose m large enough so that $d(x, x_{n_m}) < \delta/2$ and $1/n_m < \delta/2$. Thus, if $y \in B_{1/n_m}(x_{n_m})$ we have $d(y, x) \leq d(y, x_{n_m}) + d(x_{n_m}, x) < \delta/2 + \delta/2 = \delta$. Hence $B_{1/n_m}(x_{n_m}) \subseteq U_i$ which is the desired contradiction.

(v) Suppose X is compact and let $\{U_i : i \in I\}$ be a collection of open subsets $U_i \subseteq X$ with $X = \bigcup_{i \in I} U_i$. Let $\epsilon > 0$ be a Lebesgue number for this cover (as in part (iv)). Let $\{x_1, \ldots, x_n\}$ be a finite ϵ-net (as in part (iii)). Now for each k with $1 \leq k \leq n$ there exists an $i_k \in I$ with $B_\epsilon(x_k) \subseteq U_{i_k}$. So we have

$$X = \bigcup_{i=1}^{k} B_\epsilon(x_i) \subseteq \bigcup_{i=1}^{k} U_{i_k} \subseteq X$$

and hence $X = \bigcup_{i=1}^{k} U_{i_k}$.