1 Show that for any complex z,
(a) $\cos^2 z + \sin^2 z = 1$,
(b) $\sin(2z) = 2\sin z \cos z$

2 (a) By writing $\cos z = \frac{e^{iz}}{2}(1 + e^{-2iz})$ or otherwise, determine all complex z for which $\cos z = 0$.
(b) Solve the equation $\cosh z = 0$ in complex numbers.
(c) Solve the equation $\sin z + \cos z = 0$ in complex numbers.
(d) Solve the equation $e^{\frac{1}{z}} = \frac{e^2}{\sqrt{2}}(1 + i)$.

3 Write each of the following in $x + iy$ form:
(a) $4e^{i\pi/3} + \sqrt{2}$, (b) $\cos i$, (c) $\sin(\pi/2 + 2i)$, (d) $\sinh(i\pi/2)$, (e) $\sinh i + \cosh i$.

4 The real axis and the imaginary axis divide $\mathbb{C}$ into four quadrants as follows:

$$\begin{array}{c|c} \Omega_2 & \Omega_1 \\ \hline \Omega_3 & \Omega_4 \end{array}$$

By considering the modulus of the function, determine the images of $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ under the exponential map $z \mapsto e^z$.

5 Using the principal branch of $\log z$, determine the $x + iy$ form of
(a) $\log(2i)$, (b) $\sqrt{2i}$, (c) i^i.

6 Give examples to illustrate that, in general, for complex numbers z, w,
(a) $\log e^z \neq z$,
(b) $\log(zw) \neq \log z + \log w$,
(c) $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$.
Here all of the functions are defined with the principal branch of $\log z$.

7 (a) Determine $1^{1/4}$ if z^w is defined using the principal branch of logarithm.
(b) What are the other possible values of $1^{1/4}$ if the branch is not principal?

8 (a) Determine the value of $\sqrt{(2i)}$ according to the following three branches of the $\log$ function: (i) the principal branch; (ii) $\pi < \arg z < 3\pi$; (iii) $4\pi < \arg z < 6\pi$.
(b) For any non-zero $z = re^{i\phi}$ and any branch of $\log z$ for which $\sqrt{z}$ is defined show that either $\sqrt{z} = \sqrt{r}e^{i\phi/2}$ or $\sqrt{z} = -\sqrt{r}e^{i\phi/2}$.
(c) More generally, for non-zero $z = re^{i\phi}$ and an integer $n \geq 1$ show that there are exactly n possible "n-th roots of z", that is values of $z^{1/n}$ for various choices of the branch of $\log z$.

9 (a) From the definition of complex differentiability, show that $f(z) = 1/z$ is complex differentiable for all non-zero complex z, and determine its derivative.
(b) Verify the Cauchy-Riemann equations for $f(z) = 1/z$.

10 (a) Prove that $f(z) = |z|$ is not complex differentiable anywhere.
(b) Show that $g(z) = z\overline{z} = |z|^2$ is differentiable at the origin and nowhere else. Find $g'(0)$.

11 Find out where the following functions are differentiable and give formulae for their derivatives (from the lectures we already know that exp, trigonometric functions and polynomials are differentiable everywhere):

(a) $\frac{z\cos z}{1+z^2}$; (b) $\frac{e^z}{z}$; (c) $\frac{e^z+1}{e^z-1}$; (d) $\frac{\cos z}{\cos z+\sin z}$.

12 Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there. [Hint: consider what happens as $z \to 0$ along the line $y = x$ and the line $y = 0$.]

13 At which points are the following functions differentiable?
(i) $f(z) = x^2 + 2ixy$;
(ii) $f(z) = 2xy + i(x + \frac{2}{3}y^3)$;
(iii) $f(z) = x \cosh y + \sin(iy) \cos x$;
(iv) $f(z) = e^{-1/|z|^2}$ $(z \neq 0)$, $f(0) = 0$.

14 Find all complex differentiable functions defined on the whole of $\mathbb{C}$ of the form $f(z) = u(x) + iv(y)$ where u and v are both real valued.

15 Show that the principle branch of the complex logarithm function is complex differentiable at all points of $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, and has derivative $1/z$. [Hint: notice that if $z = x + iy \neq 0$ we can write

$$\operatorname{Arg}(z) = \begin{cases} \arctan(y/x) & \text{if} \quad x > 0, \\ \arctan(y/x) + \operatorname{sgn}(y)\pi & \text{if} \quad x < 0, y \neq 0, \\ \operatorname{sgn}(y)\pi/2 & \text{if} \quad x = 0, y \neq 0, \end{cases}$$

where $\operatorname{sgn}(y)$ is the standard sign function taking values ± 1 depending on whether y is strictly positive or strictly negative.]

16 Are the following functions $f(z) = f(x+iy)$ complex differentiable? [Remember to justify your responses.] For those that are, determine the derivative $f'(z)$.
(a) $f(z) = \dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2}\, i, \qquad (z \neq 0)$
(b) $f(z) = \sin(y) + i\cos(x)$,
(c) $f(z) = \overline{e^{\bar{z}}}$,
(d) $f(z) = \tan(z) \quad [= \frac{\sin z}{\cos z}]$.

17 Let $f(z)$ be a holomorphic function. Prove the following variants of the **Zero derivative theorem**, which says that, if any one of the following conditions hold on a (connected) open set X of complex numbers then $f(z)$ is constant on X.
(i) $f(z)$ is a real number for all $z \in X$.
(ii) the real part of $f(z)$ is constant on X.
(iii) the modulus of $f(z)$ is constant on X.

Remark: for instance, (i) shows that the functions $|z|$, Re z, Im z and $\arg z$ are not holomorphic.