Q.1 *Show that for any complex z,*
(a) $\cos^2 z + \sin^2 z = 1$,
(b) $\sin(2z) = 2\sin z\cos z$

S.1 (a) LHS $= \dfrac{1}{4}(e^{iz}+e^{-iz})^2 + \dfrac{1}{4i^2}(e^{iz}-e^{-iz})^2 = \dfrac{1}{4}(e^{2iz}+2+e^{-2iz}) - \dfrac{1}{4}(e^{2iz}-2+e^{-2iz}) = \dfrac{1}{4}(2+2) = 1.$
(b) RHS $= \dfrac{2}{4i}(e^{iz}-e^{-iz})(e^{iz}+e^{-iz}) = \dfrac{1}{2i}(e^{2iz}-e^{-2iz}) =$ LHS.

Q.2 *(a) By writing* $\cos z = \frac{e^{iz}}{2}(1+e^{-2iz})$ *or otherwise, determine all complex z for which* $\cos z = 0$.
(b) Solve the equation $\cosh z = 0$ *in complex numbers.*
(c) Solve the equation $\sin z + \cos z = 0$ *in complex numbers.*
(d) Solve the equation $e^{\frac{1}{z}} = \frac{e^2}{\sqrt{2}}(1+i)$.

S.2 (a) $\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) = \frac{e^{iz}}{2}(1+e^{-2iz})$. Since e^{iz} is never zero, we see that $\cos z = 0$ if and only if $1+e^{-2iz} = 0$; i.e., if and only if $e^{i\pi-2iz} = 1$. From a result in class we see that this happens if and only if $i\pi - 2iz = 2m\pi i$, for $m \in \mathbb{Z}$. That is, if and only if $z = (m+\dfrac{1}{2})\pi$ for $m \in \mathbb{Z}$. Thus the only zeros of $\cos z$ in the complex plane are those we already know about on the real axis.
(b) Either do it directly using the formula, or use part (a) together with the fact that $\cosh z = \cos(iz)$. Either way the answer is that the required solutions are $z = i(m+\dfrac{1}{2})\pi$ for $m \in \mathbb{Z}$.
(c) We want to solve $0 = \sin z + \cos z = (1/2i)(e^{iz}-e^{-iz}) + (1/2)(e^{iz}+e^{-iz}) = (1/2)((e^{iz}(1-i) + e^{-iz}(1+i))$. So we need to solve $e^{2iz} = -\frac{1+i}{1-i} = -i = e^{-i\pi/2}$; that is, $e^{2iz+i\pi/2} = 1$. Hence the solutions are $2iz + i\pi/2 = 2m\pi i$, for $m \in \mathbb{Z}$, ie $z = \pi(m-\frac{1}{4})$, $m \in \mathbb{Z}$. Again, these are all on the real axis.
(d) We have $z \neq 0$, so let $w = 1/z$. Then the equation becomes $e^w = e^2\frac{1+i}{\sqrt{2}} = e^{2+i\pi/4}$; thus $e^{w-2-i\pi/4} = 1$ and $w - 2 - i\pi/4 = 2m\pi$, for $m \in \mathbb{Z}$. So, $1/z = 2 + i(2m\pi + \frac{\pi}{4})$ and $z = \frac{1}{2+i(2m\pi+\frac{\pi}{4})} = \frac{2-i(2m\pi+\frac{\pi}{4})}{4+(2m\pi+\frac{\pi}{4})^2}$ for $m \in \mathbb{Z}$.

Q.3 *Write each of the following in $x+iy$ form:*
(a) $4e^{i\pi/3}+\sqrt{2}$, *(b)* $\cos i$, *(c)* $\sin(\pi/2+2i)$, *(d)* $\sinh(i\pi/2)$, *(e)* $\sinh i + \cosh i$.

S.3 (a) $4e^{i\pi/3}+\sqrt{2} = 4\cos(\pi/3) + 4i\sin(\pi/3) + \sqrt{2} = (2+\sqrt{2})) + i2\sqrt{3}$.
(b) $\cos i = \frac{1}{2}\left(e^{ii}+e^{-ii}\right) = \frac{1}{2}\left(e+\frac{1}{e}\right) \quad (= \cosh 1)$.
(c) Using the usual trig formulae, $\sin(\pi/2+2i) = \cos(2i) = \cosh 2$.
(d) Since $\sin(iz) = i\sinh(z)$ we have $\sinh(i\pi/2) = -i\sin(-\pi/2) = i$.
(e) In general, $\sinh z + \cosh z = e^z$, so $\sinh i + \cosh i = e^i = \cos 1 + i\sin 1$.

Q.4 *The real axis and the imaginary axis divide $\mathbb{C}$ into four quadrants as follows:*

Ω_2	Ω_1
Ω_3	Ω_4

By considering the modulus of the function, determine the images of $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ *under the exponential map* $z \mapsto e^z$.

S.4 If $z = x+iy$ then $|e^z| = e^x$ and $\arg(e^z) = y$. Since $x > 0$ on Ω_1, this region is mapped to points with modulus greater than 1. In addition, all arguments are 'hit' as y ranges across the possitive reals. Thus, Ω_1 is mapped to the exterior of the unit circle. Conversely, if $w = re^{i\theta}$ $(0 < \theta \leq 2\pi, r > 1)$ is outside this circle then there exists $x > 0$ such that $e^x = r$ so that $e^{x+i\theta} = re^{i\theta}$. Since $r + i\theta$ is in Ω_1 the map is onto. Thus, the image of Ω_1 is the whole of the exterior of the unit circle (covered an infinite number of times!).

Similarly, the image of Ω_4 is also the whole of the exterior of the unit circle (covered an infinite number of times): here, $|e^z| = e^x > 1$ and $\arg(e^z) = y < 0$ since $x > 0$ and $y < 0$. All angles must again be 'hit' (except this time clockwise!). To see that the map is onto; if $w = re^{i\theta}$ $(-2\pi \leq \theta < 0, r > 1)$ we can always find an x so that $e^x = r$ so $e^{x+i\theta} = re^{i\theta}$ and $x + i\theta$ is in Ω_4.

For Ω_2, $|e^z| = e^x < 1$ and $\arg(e^z) = y > 0$. All angles are 'hit' so the image is contained in the interior of the unit circle. But, note that $e^x \neq 0$ for any x, so the origin is excluded. Thus, the image is the punctured unit disc, i.e., the area inside the unit circle without the origin. The map is onto on his region since for $w = re^{i\theta}$ $(0 < \theta \leq 2\pi, r < 1)$ there is $x < 0$ such that $e^{x+i\theta} = re^{i\theta}$ and $x + i\theta$ is in Ω_2. A similar argument works for Ω_3 (the image of which is again the punctured unit disc).

Q.5 *Using the principal branch of* $\log z$, *determine the* $x + iy$ *form of*
(a) $\log(2i)$, (b) $\sqrt{2i}$, (c) i^i.

S.5 (a) For the principle branch, $\operatorname{Log} z = \log|z| + i\operatorname{Arg}(z)$ and hence $\operatorname{Log}(2i) = \log 2 + i(\pi/2)$.
(b) $\sqrt{2i} = (2i)^{1/2} = e^{(1/2)\operatorname{Log}(2i)} = e^{(1/2)\log 2}e^{(1/2)i(\pi/2)} = \sqrt{2}\,e^{i\pi/4} = 1 + i$.
(c) $i^i = e^{i\operatorname{Log} i} = e^{ii\pi/2} = e^{-\pi/2}$.

Q.6 *Give examples to illustrate that, in general, for complex numbers* z, w,
(a) $\log e^z \neq z$,
(b) $\log(zw) \neq \log z + \log w$,
(c) $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$.
Here all of the functions are defined with the principal branch of $\log z$.

S.6 As usual, work with the principal branch of $\log z$ - the same examples work for every branch.
(a) For example, take $z = 2\pi i$. Then LHS $= \operatorname{Log} 1 = 0 \neq 2\pi i$.
(b) We want two points that 'cross the branch cut' when multiplied. For example, we could take $z = w = (-1+i)/\sqrt{2}$. Then $zw = -i$ and so LHS $= \operatorname{Log}(-i) = \log 1 - i(\pi/2) = -i\pi/2$. But $\operatorname{Log} z = \operatorname{Log} w = \log 1 + i3\pi/4 = i3\pi/4$ so RHS$= 2(i3\pi/4) = i3\pi/2$.
(c) Again, we could take $z = w = (-1+i)/\sqrt{2}$. Then $\sqrt{zw} = e^{(1/2)\log(-i)} = e^{-(1/2)i(\pi/2)} = e^{-i(\pi/4)} = (1-i)/\sqrt{2}$. But, $\sqrt{z} = \sqrt{w} = e^{(1/2)(i3\pi/4)} = e^{i3\pi/8}$ and so RHS$= e^{i3\pi/8}e^{i3\pi/8} = e^{i3\pi/4} = (-1+i)/\sqrt{2}$. [Note that the RHS recovers z (and so w), but the former gives a rotation of z by angle π. Therefore, we still have $(\sqrt{zw})^2 = (\sqrt{z}\sqrt{w})^2$.]

Q.7 *(a) Determine* $1^{1/4}$ *if* z^w *is defined using the principal branch of logarithm.*
(b) What are the other possible values of $1^{1/4}$ *if the branch is not principal?*

S.7 (a) Using the principle branch $1^{1/4} = e^{(1/4)\operatorname{Log} 1} = e^{(1/4)(\log 1 + i0)} = e^0 = 1$.
(b) For other branches of log, say the kth branch $\log_k(z) = \log|z| + i(\operatorname{Arg}(z) + 2k\pi)$, we have $\log_k(1) = \log 1 + i(0 + 2k\pi) = i2k\pi$. Thus the possible values of $1^{1/4}$ are $= \exp((1/4)\log_k(1)) = e^{(1/4)i2k\pi} = e^{ik\pi/2}$. This gives $1^{1/4} = \pm 1, \pm i$ (by taking $k = 0, 1, 2, 3$), as expected.

Q.8 *(a) Determine the value of* $\sqrt{(2i)}$ *according to the following three branches of the* $\log$ *function: (i) the principal branch; (ii)* $\pi < \arg z < 3\pi$*; (iii)* $4\pi < \arg z < 6\pi$.
(b) For any non-zero $z = re^{i\phi}$ *and any branch of* $\log z$ *for which* $\sqrt{z}$ *is defined show that either* $\sqrt{z} = \sqrt{r}e^{i\phi/2}$ *or* $\sqrt{z} = -\sqrt{r}e^{i\phi/2}$.
(c) More generally, for non-zero $z = re^{i\phi}$ *and an integer* $n \geq 1$ *show that there are exactly* n *possible "*n*-th roots of* z*", that is values of* $z^{1/n}$ *for various choices of the branch of* $\log z$.

S.8 (a)(i) Using the principle branch, $\sqrt{(2i)} = (2i)^{1/2} = e^{(1/2)\operatorname{Log}(2i)} = e^{(1/2)(\log 2 + i(\pi/2)} = e^{(1/2)\log 2}e^{i\pi/4} = \sqrt{2}e^{i\pi/4}\left[= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i\right]$.
(ii) This is the 1st branch of log, so $\log_1(z)$. Similar to part (i), but this time $\log_1(2i) = \log|2i| + i\operatorname{Arg}(2i) + i2\pi = \log 2 + i\pi/2 + i2\pi = \log 2 + i5\pi/2$, so the answer is $e^{(1/2)\log_1(2i)} = \sqrt{2}e^{i5\pi/4} = -1 - i$.
(iii) Note that this is not a branch defined in lectures. Here we choose arguments between 4π and 6π, so the branch cut chosen is the positive real axis. We have $\arg(2i) = 9\pi/2$ (the unique angle corresponding to $2i$ that is between 4π and 6π). Thus, defining $\log(z) := \log|z| + i\arg(z)$ in the natural way for this branch cut, the answer is $e^{(1/2)\log(2i)} = \sqrt{2}e^{i9\pi/4} = \sqrt{2}e^{i\pi/4} = 1 + i$ (as in part (a)(i)).

(b) Let $z = re^{i\phi}$. Then $|z| = r$ and $\arg z = \phi + 2\pi k$ for some $k \in \mathbb{Z}$, depending on the branch of $\log z$ used. Then for any branch

$$\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\log r + i\arg z)} = e^{\frac{1}{2}\log r} e^{\frac{1}{2}i(\phi+2\pi k)} = \sqrt{r}e^{\frac{i\phi}{2}} e^{i\pi k} = \pm\sqrt{r}e^{\frac{i\phi}{2}},$$

depending on whether k is even or odd [since $e^{ik\pi} = (e^{i\pi})^k = (-1)^k$].
(c) Similarly to (b),

$$z^{1/n} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\log r + i\arg z)} = e^{\frac{1}{n}\log r} e^{\frac{1}{n}i(\phi+2\pi k)} = r^{\frac{1}{n}} e^{\frac{i\phi}{n}} e^{\frac{2i\pi k}{n}}.$$

Using that $e^{z_1} = e^{z_2}$ if and only if $z_1 - z_2 = i2\pi m$ (for some integer $m \in \mathbb{Z}$), we see that $r^{\frac{1}{n}} e^{\frac{i\phi}{n}} e^{\frac{i2\pi k_1}{n}} = r^{\frac{1}{n}} e^{\frac{i\phi}{n}} e^{\frac{i2\pi k_2}{n}}$ if and only if $\frac{i2\pi k_1}{n} - \frac{i2\pi k_2}{n} = i2\pi m$, which holds if and only if $k_1 - k_2 = mn$ is a multiple of n. It follows that there are exactly n distinct values of $z^{1/n}$, namely $\sqrt[n]{r} \cdot exp\left(\frac{i\phi}{n} + \frac{i2\pi k}{n}\right)$ for $k = 0, \ldots, n-1$, (for instance).

Q.9 *(a) From the definition of complex differentiability, show that $f(z) = 1/z$ is complex differentiable for all non-zero complex z, and determine its derivative.*
(b) Verify the Cauchy-Riemann equations for $f(z) = 1/z$.

S.9 (a) We have to investigate $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$. For $z \neq 0$ we have

$$\lim_{h\to 0} \frac{1/(z+h) - (1/z)}{h} = \lim_{h\to 0} -\frac{h/(z(z+h))}{h} = \lim_{h\to 0} -\frac{1}{z(z+h)} = -\frac{1}{z^2}.$$

Thus $f(z) = 1/z$ is complex differentiable for all non-zero complex z, and its derivative is $-1/z^2$.
(b) For $f(z) = 1/z = (x - iy)/(x^2 + y^2)$, it is straightforward to check that $u(x, y) = x/(x^2 + y^2)$ and $v(x, y) = -y/(x^2 + y^2)$ satisfy the Cauchy-Riemann equations at all points other than $x = y = 0$. For example,

$$u_x = \frac{1 \cdot (x^2 + y^2) - x \cdot (2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \text{ and } v_y = \frac{-1 \cdot (x^2 + y^2) - (-y) \cdot (2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Q.10 *(a) Prove that $f(z) = |z|$ is not complex differentiable anywhere.*
(b) Show that $g(z) = z\overline{z} = |z|^2$ is differentiable at the origin and nowhere else. Find $g'(0)$.

S.10 (a) $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = \sqrt{x^2 + y^2}$ and $v(x, y) = 0$. Thus, for $z \neq 0$, $u_x = \frac{x}{\sqrt{x^2+y^2}}$, $u_y = \frac{y}{\sqrt{x^2+y^2}}$, $v_x = 0$ and $v_y = 0$. Thus $f(z)$ is not complex differentiable for $z \neq 0$, since both Cauchy-Riemann equations do not hold at any of these points. [Alternatively, if you don't want to use the C-R equations, we can calculate the limits explicitly from the real and imaginary directions:

$$\lim_{h\to 0, h\in\mathbb{R}} \frac{f(z+h) - f(z)}{h} = ... = u_x(x, y) + iv_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}},$$
$$\lim_{h\to 0, h\in\mathbb{R}} \frac{f(z+ih) - f(z)}{ih} = ... = \frac{1}{i}(u_y(x, y) + iv_y(x, y)) = -\frac{iy}{\sqrt{x^2 + y^2}}.$$

These never match (for $z \neq 0$) so the limit in the definition of complex differentiability doesn't exist. (Of course, this is how we derived the C-R equations in the first place - it is the same thing).]

Notice finally that the partial derivatives don't exist at $z = 0$, so it cannot be complex differentiable there. Full details: for example, the real limit

$$u_x(0, 0) := \lim_{h\to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

does not exist, since the limit from the RHS ($h > 0$) is 1, while from the LHS ($h < 0$) it is -1.

Thus $f(z)$ is not complex differentiable anywhere.

(b) Here $u(x, y) = x^2 + y^2$ while $v(x, y) = 0$. Now

$$u_x = 2x, \quad v_y = 0, \quad u_y = 2y, \quad v_y = 0,$$

and the Cauchy-Riemann equations only hold when $x = 0$ and $y = 0$; that is, where $z = 0$. So g is not complex differentiable outside the origin.

At the origin we have

$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h\bar{h} - 0}{h} = \lim_{h\to 0} \bar{h} = 0.$$

Therefore g is differentiable at the origin with derivative $g'(0) = 0$. [Alternatively, note that the partial derivatives are continuous at all points, so it follows from Theorem 3.5 that g is differentiable at $z = 0$, since the Cauchy-Riemann equations hold there. We then have $g'(0) = u_x(0, 0) + iv_x(0, 0) = 0$.]

Q.11 *Find out where the following functions are differentiable and give formulae for their derivatives (from the lectures we already know that exp, trigonometric functions and polynomials are differentiable everywhere):*

(a) $\frac{z \cos z}{1+z^2}$; *(b)* $\frac{e^z}{z}$; *(c)* $\frac{e^z+1}{e^z-1}$; *(d)* $\frac{\cos z}{\cos z+\sin z}$.

S.11 These are all combinations of standard complex functions that we already know are complex differentiable. Hence, by the product/quotient rules they will be complex differentiable at all points where they are defined (so wherever the denominators are non-zero). Their derivatives will then be given by the expected formulae:

(a) The function is complex differentiable everywhere except $z = \pm i$. The derivative (by the product/quotient rule) is $\frac{\cos z - z \sin z}{1+z^2} - \frac{2z^2 \cos z}{(1+z^2)^2}$.

(b) Complex differentiable everywhere but $z = 0$ with derivative $\frac{e^z}{z}(1 - \frac{1}{z})$.

(c) The function is defined, and hence is differentiable, at all points except $z = i2n\pi$, $n \in \mathbb{Z}$ (by Proposition 1.9 part 3.). Moreover, using the quotient rule, the derivative is $-\frac{2e^z}{(e^z-1)^2}$.

(d) It was verified on Sheet 1 Q17c that the denominator is zero precisely when $z = \pi(m - 1/4)$ for $m \in \mathbb{Z}$. For every z outside this set, the derivative exists and simplifies to $\frac{-1}{1+2\cos z \sin z} = \frac{-1}{1+\sin(2z)}$.

Q.12 *Define* $f : \mathbb{C} \to \mathbb{C}$ *by* $f(0) = 0$, *and*

$$f(z) = \frac{(1 + i)x^3 - (1 - i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f *satisfies the Cauchy-Riemann equations at* 0 *but is not differentiable there. [Hint: consider what happens as* $z \to 0$ *along the line* $y = x$ *and the line* $y = 0$*.]*

S.12 First check the Cauchy-Riemann equations. We have

$$u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0), \quad u(0, 0) = 0,$$
$$v(x, y) = \frac{x^3 + y^3}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0), \quad v(0, 0) = 0.$$

Therefore at $z = 0$ we have

$$\begin{aligned}
u_x(0,0) &:= \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x^3-0}{x(x^2+0)} = 1\\
v_x(0,0) &:= \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x^3+0}{x(x^2+0)} = 1\\
u_y(0,0) &:= \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-y^3}{y(0+y^2)} = -1\\
v_y(0,0) &:= \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0+y^3}{y(0+y^2)} = 1
\end{aligned}$$

and the Cauchy-Riemann equations hold.

But, now consider the limit definition of differentiability. A point h on the line $y = x$ is always of the form $h = k + ik$ (for k real), so by letting z approach 0 along the line $x = y$ we have that

$$\lim_{\substack{h\to 0\\ h=k+ik}} \frac{f(0+h)-f(0)}{h} = \lim_{k\to 0} \frac{f(k+ik)-f(0)}{k+ik} = \lim_{k\to 0} \frac{(1+i)k^3-(1-i)k^3}{(k+ik)(k^2+k^2)} = \frac{2i}{2(1+i)} = \frac{1+i}{2}.$$

However, approaching along the real axis ($y = 0$) we have

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(1+i)h^3-(1-i)0^3}{(h)(h^2+0^2)} = 1+i,$$

so the limit does not exist and the f isn't differentiable at the origin. [Alternatively, notice that if f was differentiable at $z = 0$, then its derivative there would be equal to $f'(0) = u_x(0,0) + iv_x(0,0) = 1 + i$, by Proposition 3.3, which does not match $\frac{1+i}{2}$.]

[Extra remark: note that this does not contravene the results of the lectures because the partials are not continuous at the origin: For example $u_x(x,y) = \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$ has limit $3/2$ when approaching the origin on the diagonal, which is not the same as $u_x(0,0) = 1$.]

Q.13 *At which points are the following functions differentiable?*
(i) $f(z) = x^2 + 2ixy$;
(ii) $f(z) = 2xy + i(x + \frac{2}{3}y^3)$;
(iii) $f(z) = x\cosh y + \sin(iy)\cos x$;
(iv) $f(z) = e^{-1/|z|^2}$ *(*$z \neq 0$*)*, $f(0) = 0$.

S.13 (i) Use Theorem 3.5 (valid as the partial derivatives are continuous); $u_x = 2x, v_y = 2x, u_y = 0, v_x = 2y$. The first C-R equation is always satisfied, the second is satisfied only when $y = 0$. Thus $f(z)$ is complex differentiable on the real axis, and nowhere else.
(ii) Here $u(x,y) = 2xy$ and $v(x,y) = x + \frac{2}{3}y^3$ and we have $u_x = 2y, v_y = 2y^2, u_y = 2x, v_x = 1$. Since the partial derivatives exist and are continuous everywhere, it follows from Theorem 3.5 that $f(z)$ is complex differentiable precisely where the C-R equations both hold. So, f is complex differentiable at $z = x + iy$ if and only if $(u_x =)2y = 2y^2 (= v_y)$ and $(u_y =)2x = -1 (= -v_x)$. Hence, f is complex differentiable when $x = -1/2$ and y is 0 or 1; in other words, f is complex differentiable precisely at the points $-1/2$ and $-1/2 + i$, and nowhere else.
(iii) Here, since $\sin(iy) = i\sinh y$ we have $u(x,y) = x\cosh y$, $v(x,y) = \sinh y\cos x$. We have

$$u_x = \cosh y, \quad v_y = \cos x\cosh y, \quad u_y = x\sinh y, \quad v_x = -\sin x\sinh y.$$

The first CR-equation holds only when $\cos x = 1$, so $x = 2n\pi$ (for some $n \in \mathbb{Z}$). The second C-R equation then holds when $\sinh y = 0$ or when $x = \sin x$; that is, when $y = 0$ or $2n\pi = \sin(2n\pi)$; that is, when $y = 0$ or $n = 0$. Thus, the C-R equations hold on the imaginary axis (since they hold for all y when $n = 0$) and at $z = 2n\pi$ for non-zero $n \in \mathbb{Z}$ (since they hold only for $y = 0$ when $n \neq 0$). The partials are cts so

Theorem 3.5 implies the function is complex differentiable at those points only.
(iv) For $z \neq 0$ we have $u(x,y) = e^{-1/(x^2+y^2)}$ and $v(x,y) = 0$. We have $v_y = v_x = 0$ and

$$u_x = \frac{2x}{(x^2+y^2)^2 \exp(1/(x^2+y^2))}, \quad \text{and} \quad u_y = \frac{2y}{(x^2+y^2)^2 \exp(1/(x^2+y^2))}.$$

These never satisfy the C-R equations so f is not complex differentiable for $z \neq 0$. When $z = 0$ we have

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/(h\bar{h})}}{h} = \lim_{h\to 0} \frac{\bar{h}e^{-1/(h\bar{h})}}{h\bar{h}} = \lim_{h\to 0} \bar{h} \cdot \lim_{t\to 0} \left(\frac{1}{t}\right) e^{-1/t} = 0 \cdot 0 = 0,$$

since exponential grows much faster than linear. Thus f is complex differentiable at the origin with derivative $f'(0) = 0$.

Q.14 *Find all complex differentiable functions defined on the whole of $\mathbb{C}$ of the form $f(z) = u(x) + iv(y)$ where u and v are both real valued.*

S.14 We assume that f is defined and complex differentiable on the whole complex plane. The Cauchy-Riemann equations become:

$$u_x(x) = v_y(y), \quad 0 = 0.$$

Since the left hand side of the first equation is a function of x alone, whereas the right hand side is a function of y alone, they must be constant. In other words, there must be a real number a such that $u_x(x) = v_y(y) = a$ (since, for example, u_x cannot change as y varies and vice versa). This implies that $u(x) = ax + c$, $v(y) = ay + d$ for real numbers c, d. Therefore $f(z) = az + b$, where $a \in \mathbb{R}$ and $b = c + id \in \mathbb{C}$.

Q.15 *Show that the principle branch of the complex logarithm function is complex differentiable at all points of $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, and has derivative $1/z$. [Hint: notice that if $z = x + iy \neq 0$ we can write*

$$\mathit{Arg}(z) = \begin{cases} \arctan(y/x) & \mathit{if} \quad x > 0, \\ \arctan(y/x) + \operatorname{sgn}(y)\pi & \mathit{if} \quad x < 0, y \neq 0, \\ \operatorname{sgn}(y)\pi/2 & \mathit{if} \quad x = 0, y \neq 0, \end{cases}$$

where $\operatorname{sgn}(y)$ is the standard sign function taking values ± 1 depending on whether y is strictly positive or strictly negative.]

S.15 Let $z = x + iy$. For the principle branch we have $\operatorname{Log} z = \log\sqrt{x^2+y^2} + i\operatorname{Arg}(z) = u + iv$. We will show that the partial derivatives exist and are continuous everywhere on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, and that they also satisfy the C-R equations.

For all of the domain we have

$$u_x(x,y) = \frac{\frac{\partial}{\partial x}(x^2+y^2)^{1/2}}{(x^2+y^2)^{1/2}} = \frac{\frac{1}{2}2x(x^2+y^2)^{-1/2}}{(x^2+y^2)^{1/2}} = \frac{x}{x^2+y^2}, \qquad u_y(x,y) = \frac{y}{x^2+y^2}.$$

The function $\operatorname{sgn}(y)$ is differentiable when $y \neq 0$ (with derivative zero), so for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with $x \neq 0$ we have (in both of the situations '$x > 0$' or '$x < 0, y \neq 0$')

$$v_x(x,y) = \frac{1}{1+\frac{y}{x}^2}\left(\frac{-y}{x^2}\right) + 0 = \frac{-y}{x^2+y^2}, \qquad v_y(x,y) = \frac{1}{1+\frac{y}{x}^2}\left(\frac{1}{x}\right) + 0 = \frac{x}{x^2+y^2}.$$

Thus, the C-R equations are satisfied, and the partial derivatives are continuous, when $x \neq 0$. Furthermore, when $x = 0$ (and so $y \neq 0$) we have by l'Hopital that

$$\lim_{h\to 0^+} \frac{\operatorname{Arg}(h+iy) - \operatorname{Arg}(iy)}{h} = \lim_{h\to 0^+} \frac{\arctan\frac{y}{h} - \operatorname{sgn}(y)\frac{\pi}{2}}{h} = \lim_{h\to 0^+} \frac{-\frac{y}{h^2}}{1+(\frac{y}{h})^2} = \lim_{h\to 0^+} \frac{-y}{h^2+y^2} = -\frac{1}{y}.$$

Similarly,

$$\lim_{h\to 0^-} \frac{\operatorname{Arg}(h+iy)-\operatorname{Arg}(iy)}{h} = \lim_{h\to 0^-} \frac{\arctan\frac{y}{h}+\operatorname{sgn}(y)\pi-\operatorname{sgn}(y)\frac{\pi}{2}}{h} = -\frac{1}{y}.$$

Thus, $v_x(0,y) = -\frac{1}{y}$ $[= \frac{-y}{0^2+y^2}]$. In addition, it is easily verified that

$$v_y(0,y) = \lim_{h\to 0} \frac{\operatorname{Arg}(i(y+h))-\operatorname{Arg}(iy)}{h} = \lim_{h\to 0} \frac{\operatorname{sgn}(y)\frac{\pi}{2}-\operatorname{sgn}(y)\frac{\pi}{2}}{h} = 0 \quad \left[= \frac{0}{0^2+y^2}\right].$$

Hence the CR equations hold, and all partial derivatives are continuous at all points where $\operatorname{Log} z$ is defined. Furthermore,

$$\operatorname{Log}'(z) = u_x(x,y) + iv_x(x,y) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.$$

Q.16 *Are the following functions $f(z)=f(x+iy)$ complex differentiable? [Remember to justify your responses.] For those that are, determine the derivative $f'(z)$.*

(a) $f(z) = \dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2}\, i, \qquad (z \neq 0)$

(b) $f(z) = \sin(y) + i\cos(x)$,

(c) $f(z) = \overline{e^{\overline{z}}}$,

(d) $f(z) = \tan(z) \quad [= \frac{\sin z}{\cos z}]$.

S.16 (a) For $z \neq 0$ we have

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}; \quad v_y = \frac{y^2-x^2}{(x^2+y^2)^2}; \quad u_y = \frac{-2yx}{(x^2+y^2)^2}; \quad v_x = \frac{2yx}{(x^2+y^2)^2}.$$

All are continuous and the C-R equations hold, so f is differentiable for every $z \neq 0$. The derivative is $f'(z) = u_x + iv_x = (y^2 - x^2 + i2yx)/(x^2+y^2)^2$.

(b) We have

$$u_x = 0; \quad v_y = 0; \quad v_x = -\sin(x); \quad u_y = \cos(y).$$

Thus partial derivatives are continuous, but the C-R equations are only satisfied when $\cos y = \sin x$. Now $\cos y = \sin x \iff \cos y = \cos(x - \frac{\pi}{2}) \iff y = x - \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. The derivative there is $f'(z) = u_x + iv_x = -i\sin(x) = i\cos(y)$.

(c) Simply notice that we have

$$f(z) = \overline{\exp(\overline{z})} = \overline{\exp(x-iy)} = \overline{e^x(\cos(-y)+i\sin(-y))} = \overline{e^x\cos y - ie^x\sin y} = e^x\cos y + ie^x\sin y,$$

which is just e^z. So, f is just the exponential function and is complex differentiable everywhere.

(d) This is the quotient of two complex differentiable functions, so is differentiable everywhere except when $\cos z = 0$. By Sheet 1 Q17a these exceptions occur precisely when $z = (m+1/2)\pi$ for $m \in \mathbb{Z}$. Indeed, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{e^{iz}}{2}(1 + e^{-2iz})$. Since e^{iz} is never zero, we see that $\cos z = 0$ if and only if $1 + e^{-2iz} = 0$; i.e., if and only if $e^{i\pi - 2iz} = 1$. From Proposition 1.9 (part 3.) in lectures we see that this happens if and only if $i\pi - 2iz = 2m\pi i$, for $m \in \mathbb{Z}$. That is, if and only if $z = (m+1/2)\pi$ for $m \in \mathbb{Z}$. Thus the only zeros of $\cos z$ in the complex plane are those we already know about on the real axis. The derivative elsewhere is $f'(z) = 1/(\cos z)^2$ by the quotient rule.

Q.17 *Let $f(z)$ be a holomorphic function. Prove the following variants of the* **Zero derivative theorem**, *which says that, if any one of the following conditions hold on a (connected) open set X of complex numbers then $f(z)$ is constant on X.*

(i) $f(z)$ is a real number for all $z \in X$.

(ii) the real part of $f(z)$ is constant on X.

(iii) the modulus of $f(z)$ is constant on X.

Remark: for instance, (i) shows that the functions $|z|$, Re z, Im z and $\arg z$ are not holomorphic.

S.17 Write $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are real.

(i) If $v(x,y) = 0$, the C-R equations show that $u_x = v_y = 0$ and $u_y = -v_x = 0$. Then, $u(x,y)$ and $v(x,y)$ must be constant, and hence $f(z)$ is constant on X.

(ii) The C-R equations show that $u_x = v_y = 0$ and $u_y = -v_x = 0$. Again, $u(x,y)$ and $v(x,y)$ must be constant, and hence $f(z)$ is constant on X.

(iii) Differentiating $|f|^2 = u^2 + v^2 = c$, with respect to x and y separately, we find that $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. In other words

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Either the only solution is $u = v = 0$ (and so f is constant), or the determinant of the left hand matrix is zero. In the latter case, we have $u_x v_y - v_x u_y = 0$. By the C-R equations we therefore have $u_x^2 + u_y^2 = 0$, the only solution of which is $u_x = u_y = 0$. Thus u is constant. Since we are assuming $u^2 + v^2 = c$ (is constant), we now see that v is also constant, and the result follows.