Q.1 *Let $f(z) = \left(-1 - i\sqrt{3}\right) z + 3 - 2i$. Describe f as a rotation followed by a dilation followed by a translation. Hence draw the image under f of the unit circle $|z| = 1$ and of the line $x = y$.*

S.1 Here, $f(z) = 2e^{-\frac{i2\pi}{3}} z + 3 - 2i$. Thus f represents a clockwise rotation about the origin through angle $\frac{2\pi}{3}$, followed by dilation with factor 2, followed by translation through $3 - 2i$.

So, f takes the unit circle to the circle of radius 2 centred at $3 - 2i$. We have $-2\pi/3 + \pi/4 = -5\pi/12$, so f takes the line $x = y$ to the line (with negative slope) meeting the real axis at angle $-5\pi/12$ passing through the point $3 - 2i$.

Q.2 *Show that the function $f(z) = az + b$ (with $a \neq 0$) may be described as a translation followed by a rotation followed by a dilation. Describe $f(z) = \sqrt{2}\,(1 - i)\, z + 2 - 4i$ in this way.*

S.2 We have $az + b = a(z + (b/a))$, so $f(z) = az + b$ can indeed be described as a translation through b/a, followed by a rotation about the origin by angle $\mathrm{Arg}(a)$, followed by a dilation with factor $|a|$.

For $f(z) = \sqrt{2}\,(1 - i)\, z + 2 - 4i$ we have

$$\begin{aligned} f(z) &= \sqrt{2}\,(1 - i)\, z + 2 - 4i \\ &= \sqrt{2}(1 - i)\left(z + \frac{2 - 4i}{\sqrt{2}(1 - i)}\right) = 2e^{-\frac{i\pi}{4}}\left(z + \frac{3 - i}{\sqrt{2}}\right). \end{aligned}$$

So f is translation through $\frac{3-i}{\sqrt{2}}$, followed by rotation about the origin through an angle $-\frac{\pi}{4}$ anticlockwise followed by dilation with factor 2.

Q.3 *In what subset of the complex plane is $4z^3 - 3iz^2 + 4 - 3i$ conformal?*

S.3 Let $f(z) = 4z^3 - 3iz^2 + 4 - 3i$; then $f'(z) = 12z^2 - 6iz = 6z(2z - i)$. Thus for $z \neq 0, i/2$ we have $f'(z) \neq 0$ and so f is conformal in $\mathbb{C} \setminus \{0, i/2\}$, but is not conformal at 0 or $i/2$.

Q.4 *In what subset of the complex plane is $2z^3 - 3\,(1 + i)\, z^2 + 6iz$ conformal?*

S.4 Let $f(z) = 2z^3 - 3\,(1 + i)\, z^2 + 6iz$; then

$$\begin{aligned} f'(z) &= 6z^2 - 6\,(1 + i)\, z + 6i \\ &= 6\left(z^2 - (1 + i)\, z + i\right) \\ &= 6\,(z - 1)\,(z - i)\ . \end{aligned}$$

Thus for $z \neq 1, i$, we see that $f'(z) \neq 0$. Hence f is conformal in $\mathbb{C} \setminus \{1, i\}$, but not for $z \in \{1, i\}$ because $f'(z) = 0$ there.

Q.5 *In what subset of the complex plane is $\sinh z$ conformal? For every point z_0 at which the function is not conformal, give an example of two paths (lines) through z_0 such that the angle (or the orientation of the angle) between them is not preserved by $f(z)$ at z_0.*

S.5 Let $f(z) = \sinh z$, so that $f'(z) = \cosh z$. Certainly $f(z)$ is holomorphic on $\mathbb{C}$ and

$$\begin{aligned} & f'(z) = 0 \\ \iff\ & e^z + e^{-z} = 0 \\ \iff\ & e^{2z} = -1 \\ \iff\ & e^{2z} = e^{i\pi} \\ \iff\ & 2z - i\pi = i2k\pi \\ \iff\ & z = i\left(\frac{\pi}{2} + k\pi\right) \end{aligned}$$

where $k \in \mathbb{Z}$. Let

$$S = \left\{ i\left(\frac{\pi}{2} + k\pi\right) : k \in \mathbb{Z} \right\}$$

Thus $f(z)$ is conformal on $\mathbb{C} \setminus S$, but not on S.

Now take $z_0 = i\left(\frac{\pi}{2} + k\pi\right) \in S$. Using that $\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w$ and $\sinh(z_0) = i\sin(z_0/i) = i(-1)^k$, we get $\sinh(z_0 + w) = i(-1)^k \cosh w$. Consider the two lines

$$\gamma_1(t) = z_0 + t, \quad \gamma_2(t) = z_0 + it$$

for $t \in \mathbb{R}$, then these curves meet at z_0 at an angle $\pi/2$. However, the curves

$$(f \circ \gamma_1)(t) = i(-1)^k \cosh t, \quad (f \circ \gamma_2)(t) = i(-1)^k \cos t$$

meet at $f(z_0)$ at an angle π. So the angle is not preserved at z_0.

Q.6 *At which points in $\mathbb{C}$ are the following maps conformal?*
(a) $f(z) = z^3 + 2i$ *(b) $f(x+iy) = x - 3yi$*
In both cases, for every point z_0 at which the function is not conformal, give an example of two paths (lines) through z_0 such that the angle (or the orientation of the angle) between them is not preserved by $f(z)$ at z_0.

S.6 (a) $f(z) = z^3 + 2i$ is holomorphic, but $f'(z) = 3z^2 \neq 0$ only when $z \neq 0$. So f is conformal on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Any choices of two distinct lines passing through $z = 0$ works. For example, the paths $\gamma_1(t) = (t - \frac{1}{2})e^{i\theta_1}$, $\gamma_2(t) = (t - \frac{1}{2})e^{i\theta_2}$, $t \in [0,1]$, $\theta_1 > \theta_2$ intersect at the origin at angle $\theta_1 - \theta_2$. However, $(f \circ \gamma_j)(t) = (t - \frac{1}{2})^3 e^{i3\theta_j} + 2i$ for $j = 1, 2$. The angle between the images of these paths under f is therefore $3\theta_1 - 3\theta_2$ (up to a multiple of 2π) [since $(f \circ \gamma_j)'(t) = 3(t - \frac{1}{2})^2 e^{i3\theta_j}$], and so angle is not preserved.

(b) Let $f(x+iy) = x - 3yi$. We have $u_x = 1, v_y = -3, v_x = 0, u_y = 0$ so the first C-R equation is never satisfied. Therefore f is conformal nowhere.

For the second part of the question, we must check every point in the complex plane. Let $z_0 = x_0 + iy_0$ be a complex number. Many choices of lines through z_0 work, but let's choose the simplest. Let $\gamma_1(t) = z_0 + t$ and let $\gamma_2(t) = z_0 + it$. The angle between γ_1 and γ_2 at z_0 is clearly $\pi/2$ anticlockwise (since $\gamma_1'(t) = 1$ and $\gamma_2'(t) = i$).

But, $(f \circ \gamma_1)(t) = (x_0 + t) - 3iy_0$, so $(f \circ \gamma_1)'(t) = 1$, while $(f \circ \gamma_2)(t) = x_0 - 3i(y_0 + t)$, so $(f \circ \gamma_2)'(t) = -3i$. The angle between $(f \circ \gamma_1)$ and $(f \circ \gamma_2)$ at z_0 is therefore $\pi/2$ clockwise. Hence orientation is not preserved.

Q.7 *Let $f(z) = z^2 + 2z$. Show that f is conformal at $z = i$ and describe the effect of $f'(z)$ on the tangent vectors of curves passing through this point.*

S.7 First note

$$f'(z) = 2z + 2$$
$$f'(i) = 2 + 2i \neq 0$$

so that $f(z)$ is conformal at $z = i$. Now, we have $\text{Arg}(f'(i)) = \dfrac{\pi}{4}$, so the tangent vectors at $z = i$ are rotated by $\dfrac{\pi}{4}$ anticlockwise. The dilation factor is

$$|f'(i)| = \sqrt{4+4} = 2\sqrt{2}.$$

Q.8 *Let $f(z) = 2z^3 + 3z^2$. Show that f is conformal at $z = i$ and describe the effect of $f'(z)$ on the tangent vectors of curves passing through this point.*

S.8 First note

$$f'(z) = 6z^2 + 6z$$
$$f'(i) = -6 + 6i \neq 0$$

so that $f(z)$ is conformal at $z = i$. We have $\text{Arg}(f'(i)) = \frac{3\pi}{4}$, so the tangent vectors at $z = i$ are rotated by $\frac{3\pi}{4}$ anticlockwise. The dilation factor is

$$|f'(i)| = \sqrt{6^2 + 6^2} = 6\sqrt{2}.$$

Q.9 *Is the following true or false? If f, g are conformal at a point z_0 then $f + g$ is conformal at z_0. Give a proof or a counter-example.*

S.9 False. For example, let $f(z) = z + z^2$, $g(z) = -z$ and $z_0 = 0$. Then $f'(0) = 1$, $g'(0) = -1$ so both f and g are conformal at $z = 0$. But $f(z) + g(z) = z^2$, which is not conformal at $z = 0$ because its derivative is 0 there.

Q.10 *Let $f(z) = \overline{g(z)}$ with $g(z)$ holomorphic (such functions f we call* **anti-holomorphic***). Describe geometrically what happens to tangent vectors of paths passing through a point under the map f. Conclude that f is angle-preserving, but reverses the orientation.*

S.10 Let $\gamma(t)$ be a path in $\mathbb{C}$. Then

$$\tfrac{\partial}{\partial t} f(\gamma(t)) \;=\; \tfrac{\partial}{\partial t}\overline{g(\gamma(t))} \;\overset{(*)}{=}\; \overline{\tfrac{\partial}{\partial t} g(\gamma(t))} \;=\; \overline{g'(\gamma(t)) \cdot \gamma'(t)}.$$

To see that the equality $(*)$ holds in the above, either observe that conjugation is preserved by linear maps, or explicitly notice:

$$\left[\overline{(g \circ \gamma)(t)}\right]' \;=\; [u(\gamma(t)) - iv(\gamma(t))]' \;=\; [u(\gamma(t))]' - i\,[v(\gamma(t))]' = \overline{[u(\gamma(t))]' + i\,[v(\gamma(t))]'} = \overline{(g \circ \gamma)'(t)},$$

where $g = u + iv$.

Hence tangent vectors get rotated then dilated by the holomorphic derivative $g'(z)$ of g as usual, but then they are conjugated (so reflected in the real axis). In the last step the angle is preserved, but orientation is reversed.

Q.11 *Let $f : \mathcal{D} \to \mathcal{D}'$ be a biholomorphic map between two domains $\mathcal{D}$ and $\mathcal{D}'$. By considering the equation $f(f^{-1}(w)) = w$ (for $w \in \mathcal{D}'$), show that f is conformal.*

S.11 Since f is holomorphic with holomorphic inverse, we have that $f \circ f^{-1} : \mathcal{D}' \to \mathcal{D}'$ is trivially holomorphic. Taking the derivative of both sides of the equation $f(f^{-1}(w)) = w$, with respect to w, it follows from the chain rule that $f'(f^{-1}(w)) \cdot (f^{-1})'(w) = 1$. Since f^{-1} is onto this means $f'(z) \neq 0$ for all $z \in \mathcal{D}$. Hence f is conformal on $\mathcal{D}$.

Q.12 *Let $\Omega := \{z \in \mathbb{C} : Re(z) > \sqrt{3}\,|Im(z)|\}$. Sketch the domain Ω and find its image $f(\Omega)$ under the map $f(z) = z^6$. Hence show that f is a biholomorphic map from Ω onto its image, and give the inverse function.*

S.12 The function is holomorphic on $\mathbb{C}$. It is easy to see that $\Omega = \{z \in \mathbb{C} : -\pi/6 < \text{Arg}(z) < \pi/6\}$ (sketch of this sector is omitted). If $z = |z|e^{i\,\text{Arg}(z)} \in \Omega$ then $f(z) = |z|^6 e^{i6\,\text{Arg}(z)}$. Since $\text{Arg}(z) \in (-\pi/6, \pi/6)$ we have $6\,\text{Arg}(z) \in (-\pi, \pi)$. Thus, the image of Ω under f is $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. By considering the modulus-argument form of any point in the image, it is clear that this map is injective and surjective. We know that the inverse map $z^{1/6} = \exp(\frac{1}{6}\,\text{Log}(z))$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Thus

$$z^6 : \Omega \xrightarrow{\sim} \mathbb{C} \setminus \mathbb{R}_{\leq 0}.$$