1 (i) To any matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ we associate the Möbius transformation $M_T(z) = \frac{az+b}{cz+d}$.
Given two Möbius transformations $M_T(z) = \frac{az+b}{cz+d}$ and $M_S(z) = \frac{pz+q}{rz+s}$ determine $M_T \circ M_S$ by a direct calculation and conclude

$$M_T \circ M_S = M_{TS}.$$

(ii) Show that

$$M_T = \mathrm{Id} \quad \Longleftrightarrow \quad T = t\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{(for some } t \in \mathbb{C}^*\text{)}.$$

Conclude that under composition a Möbius transformation M_T has inverse $(M_T)^{-1} = M_{T^{-1}}$.

2 Find the image of the unit circle and the unit disc under the Möbius transformation $f(z) = w = \frac{z+i}{z-i}$.

3 Find the image of the unit circle and the unit disc under the Möbius transformation $f(z) = w = \frac{(1+i)z+i-1}{iz+1}$.

4 Find the image of the unit circle and the unit disc under the transformation $f(z) = w = \frac{(1+i)z-1-i}{z+1}$.

5 Is there a Möbius transformation which maps the sides of the triangle with vertices at $-1, i$ and 1 to the sides of an equilateral triangle (all sides of equal length)? Either give an example of such a Möbius transformation, or explain why it is not possible.

6 Show that the Cayley Map $M_C = w = \frac{z-i}{z+i}$ takes the point $\frac{1}{2}(1+i)$ to the point $-\frac{1}{5}(1+2i)$. Hence, or otherwise, sketch the image of the triangle with vertices at $0, 1$ and i under M_C.

7 Find the fixed points of the inverse Cayley Map $M_{C^{-1}}$ [that is, the Möbius transformation associated with the matrix $C^{-1} = \left(\begin{smallmatrix} i & i \\ -1 & 1 \end{smallmatrix}\right)$].

8 If α and β are the two fixed points of a Möbius transformation $f(z)$, show that for all $z \neq \alpha, \beta$ and $f(z) \neq \infty$, we have

$$\frac{f(z)-\alpha}{f(z)-\beta} = K\frac{z-\alpha}{z-\beta},$$

where K is a constant.

9 Find the Möbius transformation taking the ordered set of points $\{-i, -1, i\}$ to the ordered set of points $\{-i, 0, i\}$. What is the image of the unit disc under this map? Which point is sent to ∞?

10 Find the Möbius transformation taking the ordered set of points $\{-1, 1, -i\}$ to the ordered set of points $\{1, -1, 0\}$. What is the image of the unit disc under this Möbius transformation?

11 Find the Möbius transformation taking the ordered set of points $\{-1+i,\ 0,\ 1-i\}$ to the ordered set of points $\{-1,\ -i,\ 1\}$. What is the image of the region $\mathcal{R} = \{x+iy : x+y > 0\}$ under this Möbius transformation? What happens to ∞ under this Möbius transformation?

12 Find the Möbius transformation taking the ordered set of points $\{0, 1+i, -1-i\}$ to the ordered set of points $\{1, -i, i\}$. What is the image of the region $\mathcal{R} = \{x+iy : x-y \geq 0\}$ under this Möbius transformation?

13 Let z_0 be an arbitrary complex number with $|z_0| < 1$. Show that the Möbius transformation

$$f(z) = \frac{z-z_0}{\overline{z_0}\, z - 1}$$

maps the unit disc to the unit disc, and maps z_0 to 0, and 0 to z_0. Compute $f \circ f$. What do you observe? What happens if $|z_0| = 1$? What happens if $|z_0| > 1$?