Q.1 *(i) To any matrix* $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ *we associate the Möbius transformation* $M_T(z) = \frac{az+b}{cz+d}$. *Given two Möbius transformations* $M_T(z) = \frac{az+b}{cz+d}$ *and* $M_S(z) = \frac{pz+q}{rz+s}$ *determine* $M_T \circ M_S$ *by a direct calculation and conclude*

$$M_T \circ M_S = M_{TS}.$$

(ii) Show that

$$M_T = \mathrm{Id} \qquad \Longleftrightarrow \qquad T = t\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \qquad \textit{(for some } t \in \mathbb{C}^*\textit{)}.$$

Conclude that under composition a Möbius transformation M_T *has inverse* $(M_T)^{-1} = M_{T^{-1}}$.

S.1 (i) We have $M_T \circ M_S(z) = \dfrac{(ap+br)z + (aq+bs)}{(cp+dr)z + (cq+ds)}$, but also

$$\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix} = TS.$$

(ii) $M_T = \mathrm{Id}$ means $z = M_T(z) = \frac{az+b}{cz+d}$ for every $z \in \mathbb{C}$, so $cz^2 + (d-a)z - b = 0$. Since this must hold for every z it must in particular hold for $z = 0$, and so we have $b = 0$. It must hold for $z = 1$ so $c = a - d$ and also $z = -1$ so $c = -(a-d)$. This is only possible if $c = a - d = 0$, whence $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \neq 0$ (since $\det T \neq 0$). The final claim follows immediately.

Q.2 *Find the image of the unit circle and the unit disc under the Möbius transformation* $f(z) = w = \frac{z+i}{z-i}$.

S.2 First note that $-i \mapsto 0$, $i \mapsto \infty$ and

$$1 \mapsto \frac{1+i}{1-i} = \frac{2i}{2} = i.$$

Thus the unit circle is sent to the circle or line through $0, i, \infty$; i.e. the imaginary axis. And $0 \mapsto -1$, so the unit disc is sent to the left half-plane.

Q.3 *Find the image of the unit circle and the unit disc under the Möbius transformation* $f(z) = w = \frac{(1+i)z+i-1}{iz+1}$.

S.3 First note that $i \mapsto \infty$ and

$$1 \mapsto \frac{2i}{1+i} = \frac{2i(1-i)}{2} = 1+i, \qquad -1 \mapsto \frac{-2}{1-i} = \frac{-2(1+i)}{2} = -1-i$$

so the unit circle is sent to the line or circle through $\infty, 1+i, -1-i$; in other words, the line $u = v$ in the (u,v)-plane (with angle $\pi/4$). And $0 \mapsto -1+i$, so the unit disc is sent to $\{(u,v) : v > u\}$. (Note that we could have used $-i \mapsto 0$ in place of one of the three points.)

Q.4 *Find the image of the unit circle and the unit disc under the transformation* $f(z) = w = \frac{(1+i)z-1-i}{z+1}$.

S.4 First note that $z = -1 \mapsto w = \infty$, so the unit circle is sent to a circle or line through ∞; i.e. to a line.

Also, $z = 1$ maps to $w = 0$ whilst $z = i$ maps to

$$w = \frac{(1+i)i - 1 - i}{i+1} = \frac{-2}{i+1} = -(1-i) = -1+i,$$

both of which lie on the line $u + v = 0$. Thus, the image of the unit circle must be this line. Finally, $z = 0$ is mapped to

$$w = -1-i$$

so the unit disc is sent to $\{(u,v) : u + v < 0\}$.

Q.5 *Is there a Möbius transformation which maps the sides of the triangle with vertices at* $-1, i$ *and* 1 *to the sides of an equilateral triangle (all sides of equal length)? Either give an example of such a Möbius transformation, or explain why it is not possible.*

S.5 No. A Möbius transformation is a conformal map, so preserves angles between curves. The triangle with vertices at $-1, i$ and 1 has angles $\pi/4, \pi/4, \pi/2$ so cannot be mapped to an equilateral triangle whose angles are all equal to $\pi/3$.

Q.6 *Show that the Cayley Map $M_C = w = \frac{z-i}{z+i}$ takes the point $\frac{1}{2}(1+i)$ to the point $-\frac{1}{5}(1+2i)$. Hence, or otherwise, sketch the image of the triangle with vertices at $0, 1$ and i under M_C.*

S.6 A simple calculation yields

$$M_C\left(\frac{1}{2}(1+i)\right) = \frac{\frac{1}{2}(1+i)-i}{\frac{1}{2}(1+i)+i} = \frac{1-i}{1+3i} = \frac{(1-i)(1-3i)}{10} = \frac{-2-4i}{10} = -\frac{1}{5}(1+2i).$$

The triangle in question is made up of three line segments. Each line segment must be taken to a line segment or circular arc by a Möbius transformation. Note that under M_C

$$0 \mapsto \frac{-i}{i} = -1; \quad 1 \mapsto \frac{1-i}{1+i} = -i; \quad i \mapsto 0.$$

First we check where the line segment from 0 to 1 is taken. We have

$$\frac{1}{2} \mapsto \frac{\frac{1}{2}-i}{\frac{1}{2}+i} = \frac{1-2i}{1+2i} = \frac{(1-2i)^2}{5} = \frac{-3-4i}{5},$$

which lies on the unit circle in the third quadrant. Thus the first line segment is taken to the circular arc in the third quadrant from -1 to $-i$.

Next we check the line segment from 0 to i. We have

$$\frac{i}{2} \mapsto \frac{\frac{i}{2}-i}{\frac{i}{2}+i} = \frac{-i}{3i} = -\frac{1}{3},$$

which lies on the real axis. Thus, the second line segment is taken to the line segment joining -1 and 0.

Finally, we consider the line segment between 1 and i. The point $\frac{1}{2}(1+i)$ lies on this line segment and is taken to $-\frac{1}{5}(1+2i)$. We know that the image must be a line segment or a circular arc joining 0 and $-i$, and since $-\frac{1}{5}(1+2i)$ does not lie on the imaginary axis it must be a circular arc. Indeed, it must be the circular arc between 0 and $-i$ in the third quadrant passing through 0 and $-i$.

The image of the triangle is the union of the images of the three line segments considered; starting from the origin, this is a line from 0 to -1, then a circular arc to $-i$ (following the unit circle), then a circular arc in the third quadrant joining $-i$ back to 0. So it looks like a fang!

Q.7 *Find the fixed points of the inverse Cayley Map $M_{C^{-1}}$ [that is, the Möbius transformation associated with the matrix $C^{-1} = \left(\begin{smallmatrix} i & i \\ -1 & 1 \end{smallmatrix}\right)$].*

S.7 If z_0 is a fixed point it must satisfy

$$\frac{iz_0+i}{-z_0+1} = z_0, \quad \text{and so} \quad (z_0)^2 - (1-i)z_0 + i = 0.$$

Solving this (by completing the square or otherwise) yields

$$z_0 = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right) - i\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right).$$

Q.8 *If α and β are the two fixed points of a Möbius transformation $f(z)$, show that for all $z \neq \alpha, \beta$ and $f(z) \neq \infty$, we have*

$$\frac{f(z)-\alpha}{f(z)-\beta} = K\frac{z-\alpha}{z-\beta},$$

where K is a constant.

S.8 Let z_1 be a point different from α and β, and let w_1 be the image of z_1 under the Möbius transformation. We know the Möbius transformation preserves the cross-ratio so (taking $z_2 = w_2 = \alpha$ and $z_3 = w_3 = \beta$) we have

$$\frac{(f(z)-\alpha)(w_1-\beta)}{(f(z)-\beta)(w_1-\alpha)} = \frac{(z-\alpha)(z_1-\beta)}{(z-\beta)(z_1-\alpha)}.$$

Hence

$$\frac{f(z)-\alpha}{f(z)-\beta} = K\frac{z-\alpha}{z-\beta},$$

where $K = \frac{(w_1-\alpha)(z_1-\beta)}{(w_1-\beta)(z_1-\alpha)}$.

Q.9 *Find the Möbius transformation taking the ordered set of points $\{-i, -1, i\}$ to the ordered set of points $\{-i, 0, i\}$. What is the image of the unit disc under this map? Which point is sent to ∞?*

S.9 Consider

$$z_1 = -i, z_2 = -1, z_3 = i, \qquad w_1 = -i, w_2 = 0, w_3 = i \tag{1}$$

We know that Möbius transformations preserve the cross-ratio; i.e.,

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}.$$

We simply need to solve for w in terms of z. In this case, we have

$$\begin{aligned}
\frac{(w-0)(-i-i)}{(w-i)(-i-0)} = \frac{(z+1)(-i-i)}{(z-i)(-i+1)} \quad &\iff \quad \frac{-2iw}{-i(w-i)} = \frac{-2i(z+1)}{(z-i)(-i+1)} \\
&\iff \quad w(z-i)(1-i) = -i(w-i)(z+1) \\
&\iff \quad (1-i)wz - i(1-i)w = -iwz - z - iw - 1 \\
&\iff \quad wz - w = -z - 1 \\
&\iff \quad w(z-1) = -z-1 \\
&\iff \quad w = -\frac{z+1}{z-1}.
\end{aligned}$$

Any three distinct points lie on a unique line or circle, so the Möbius transformation $w = f(z)$ taking z_i into w_i for each i must take the unit circle to the imaginary axis (as the three image points are on the imaginary axis, see (1)). Alternatively, we can take each point on the unit circle as $z = e^{it}$ and get

$$f(e^{it}) = -\frac{e^{it}+1}{e^{it}-1} = -\frac{e^{it/2}(e^{it/2}+e^{-it/2})}{e^{it/2}(e^{it/2}-e^{-it/2})} = i\frac{\cos(t/2)}{\sin(t/2)}$$

which goes along the imaginary axis if we vary $t \in (-\pi, \pi]$, so this proves again that the unit circle is mapped to the imaginary axis. Choosing $z = 0$ in the interior of the unit disc, we see that its image $f(z) = 1$ under this map is in the right half-plane, so the interior of the circle must be sent to the right half plane $\mathbb{H}_R := \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$. Note that $z = 1$ is sent to $w = \infty$.

Q.10 *Find the Möbius transformation taking the ordered set of points $\{-1, 1, -i\}$ to the ordered set of points $\{1, -1, 0\}$. What is the image of the unit disc under this Möbius transformation?*

S.10 Here is a different way of solving this type of problem via *linear algebra*: We need to find complex numbers a, b, c, d so that

$$1 = \frac{a(-1)+b}{c(-1)+d}\,, \quad ie \quad -c+d = -a+b\,. \tag{1}$$

$$-1 = \frac{a(1)+b}{c(1)+d}\,, \quad ie \quad -c-d = a+b\,. \tag{2}$$

$$0 = \frac{a(-i)+b}{c(-i)+d}\,, \quad ie \quad -ai+b=0\,. \qquad (3)$$

Three linear equations in 4 unknowns, so 1 degree of freedom. Let's choose $a = 1$. Then (3) gives $b = i$. Then $(1) + (2)$ gives $-2c = 2b$, ie, $c = -i$, so (2) then gives $d = -c - a - b$, ie $d = i - 1 - i = -1$.

So, required Möbius transformation is

$$f(z) = \frac{z+i}{-iz-1}\,.$$

You can check this by substituting the given values *[e.g., put* $z = -1$ *and get* $f(-1) = (-1+i)/(i-1) = (-1+i)(-i-1)/2 = 1$*].*

We note that if $z = -i$ then $f(-i) = 0$, so the unit circle is mapped to the real axis. Also, since $z = 0$ gives $f(0) = -i$, we see that the unit disc is mapped to the lower half-plane.

Q.11 *Find the Möbius transformation taking the ordered set of points* $\{-1+i,\ 0,\ 1-i\}$ *to the ordered set of points* $\{-1,\ -i,\ 1\}$. *What is the image of the region* $\mathcal{R} = \{x+iy : x+y > 0\}$ *under this Möbius transformation? What happens to* ∞ *under this Möbius transformation?*

S.11 We know that a Möbius transformation $w = f(z)$ preserves the cross-ratio; i.e.,

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}.$$

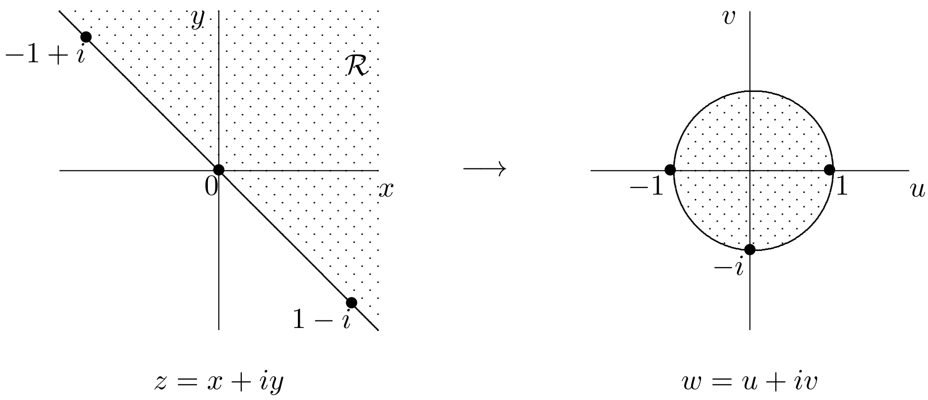
We solve for w in terms of z. In this case, the above simplifies to

$$\begin{aligned}
\frac{(w-(-i))(-1-1)}{(w-1)(-1-(-i))} &= \frac{(z-0)(-1+i-(1-i))}{(z-(1-i))(-1+i-0)} \\
-2(z-1+i)(w+i) &= (-2z+2iz)(w-1) \\
-(z-1+i)(w+i) &= z(i-1)(w-1) \\
-z+(1+i) &= (iz-1+i)w \\
w &= \frac{-z+1+i}{iz-1+i} \quad \left[= \frac{iz+1-i}{z+1+i}\right].
\end{aligned}$$

The boundary of $\mathcal{R}$ is the line $L = \{x+iy : x+y = 0\}$, and the points $-1+i, 0, 1-i$ all lie on this line. Thus, the Möbius transformation maps this line to the line or circle through $-1, -i, 1$. Clearly, this is the unit circle. We check what happens to $\mathcal{R}$:

$$z = 1+i \quad \mapsto \quad w = \frac{i(1+i)+1-i}{1+i+1+i} = 0,$$

so $\mathcal{R}$ is sent to the interior of the unit circle in the w-plane. Finally, $z = \infty$ is mapped to $w = \frac{i}{1} = i$.



Q.12 *Find the Möbius transformation taking the ordered set of points* $\{0, 1+i, -1-i\}$ *to the ordered set of points* $\{1, -i, i\}$. *What is the image of the region* $\mathcal{R} = \{x + iy : x - y \geq 0\}$ *under this Möbius transformation?*

S.12 We know that a Möbius transformation $w = f(z)$ preserves the cross-ratio; i.e.,

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

We solve for w in terms of z. In this case, the above simplifies to

$$\frac{(w + i)(1 - i)}{(w - i)(1 + i)} = -\frac{z - (1 + i)}{z + (1 + i)},$$

and the solution is

$$w = \frac{z + i - 1}{-z + i - 1}.$$

The image of the line $x = y$ is the unit circle, and, since 1 maps to $i/(i - 2)$ which clearly has modulus less than 1, the given region maps to the closed unit disc.

[Note that strictly speaking $\mathcal{R}$ *here is not a 'region/domain' since it is closed, so in this case the Möbius trans is not a biholomorphic map from* $\mathcal{R}$ *to* $\overline{B}_1(0)$ *by our definition.]*

Q.13 *Let* z_0 *be an arbitrary complex number with* $|z_0| < 1$. *Show that the Möbius transformation*

$$f(z) = \frac{z - z_0}{\overline{z_0}\, z - 1}$$

maps the unit disc to the unit disc, and maps z_0 *to* 0, *and* 0 *to* z_0. *Compute* $f \circ f$. *What do you observe? What happens if* $|z_0| = 1$? *What happens if* $|z_0| > 1$?

S.13 Assume $|z| = 1$. Then

$$|z - z_0| = |z\bar{z} - z_0\bar{z}| = |1 - z_0\bar{z}| = |\bar{z}_0 z - 1|,$$

so $|f(z)| = 1$. Hence the Möbius transformation maps the unit circle to the unit circle. Clearly, $z_0 \mapsto 0$, so the unit disc is mapped to the unit disc. We have

$$(f \circ f)(z) = \frac{\frac{z - z_0}{\overline{z_0}\, z - 1} - z_0}{\overline{z_0}\left(\frac{z - z_0}{\overline{z_0}\, z - 1}\right) - 1} = \frac{z - z_0 - z_0(\overline{z_0}\, z - 1)}{\overline{z_0}\,(z - z_0) - (\overline{z_0}\, z - 1)} = \frac{z(1 - z_0\, \overline{z_0})}{(1 - z_0\, \overline{z_0})} = z,$$

so f is its own inverse (it is an "involution").

If $|z_0| = 1$, then the determinant of the matrix associated with f is "$ad - bc$" $= -1 - (-z_0\bar{z}_0) = 0$, so we don't have a Möbius transformation. If $|z_0| > 1$, then $z_0 \mapsto 0$ so the Möbius transformation maps the exterior of the unit disc to the unit disc.