Q.1 *Is there a non-trivial Möbius transformation from the upper half-plane to itself that fixes both the points $1+i$ and $-1+i$? Either find an example of such a map or show that none exists.*

S.1 No. For example, by Proposition H2H we know that any Möbius transfomation f from the upper half-plane to itself is associated with a member of $\mathrm{SL}_2(\mathbb{R})$. That is, $f(z) = \frac{az+b}{cz+d}$ with a, b, c, d real numbers. But, for $1 + i$ and $-1 + i$ to be fixed points we must have

$$(1+i)a + b \;=\; (1+i)^2 c + (1+i)d \;=\; 2ic + (1+i)d;$$

and

$$(-1+i)a + b \;=\; (-1+i)^2 c + (-1+i)d \;=\; -2ic + (-1+i)d.$$

Adding these two equations yields $2ia + 2b = 2id$, and subtracting the second from the first yields $2a = 4ic + 2d$. Since all coefficients are real, we must have $a = d$ and $b = c = 0$ and so $f(z) = z$ is the identity map.

Q.2 *Find an automorphism of the unit disc that takes $\frac{1}{2}$ to 0 and (when considered as a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) also takes -1 to $-i$.*

S.2 A Moebius transformation f will do the trick. By Corollary D2D-improved from lectures with $z_0 = \frac{1}{2}$ we have

$$f(z) = k\left(\frac{z - \frac{1}{2}}{\frac{1}{2}z - 1}\right) = k\left(\frac{2z-1}{z-2}\right)$$

for some $k = e^{i\theta}$. Since $-1 \mapsto -i$, we have

$$-i = k\left(\frac{-2-1}{-1-2}\right) \qquad \text{and so} \qquad k = -i.$$

Thus

$$f(z) = -i\left(\frac{2z-1}{z-2}\right).$$

Q.3 *Find an automorphism of the unit disc that takes $-\frac{i}{2}$ to 0 and (when considered as a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) also takes 1 to i.*

S.3 A Moebius transformation f will do the trick. By Corollary D2D-improved from lectures with $z_0 = -i/2$ we have

$$f(z) \;=\; k\left(\frac{z - (-\frac{i}{2})}{\frac{i}{2}z - 1}\right) \;=\; k\left(\frac{2z+i}{iz-2}\right)$$

for some $k = e^{i\theta}$. Since $1 \mapsto i$, we have

$$i = k\left(\frac{2+i}{i-2}\right) \qquad \text{and so}$$

$$k = i\left(\frac{i-2}{2+i}\right) = i\left(\frac{4i-3}{5}\right) = -\frac{4+3i}{5}.$$

Thus

$$f(z) = -\left(\frac{4+3i}{5}\right)\left(\frac{2z+i}{iz-2}\right).$$

Q.4 *Find a Möbius transformation f from the upper half-plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$ that takes $1+i$ to 0 and (when considered as a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) also takes 1 to $-i$. Give an explicit formula for $f(z)$.*

S.4 The Cayley Map $M_C(z) = \frac{z-i}{z+i}$ is a Möbius transformation that maps the upper half-plane onto the unit disc, but it takes i to 0 and 1 to $-i$. Since $M_{C^{-1}}$ takes the unit circle to the half-plane we see that $g = M_{C^{-1}} \circ f$ is a Möbius transformation from $\mathbb{H}$ to itself. In other words, $f = M_C \circ g$ where g is a Möbius transfornation from $\mathbb{H}$ to itself with $g(1+i) = i$ and $g(1) = 1$. We know that all Möbius transformations that take $\mathbb{H}$ to itself are generated by invertible matrices with real entries. We need to solve

$$\frac{a(1+i)+b}{c(1+i)+d} = i, \quad \frac{a+b}{c+d} = 1$$

with real a, b, c, d such that $ad - bc \neq 0$. This is equivalent to

$$a(1+i) + b = c(-1+i) + id, \quad a + b = c + d.$$

Taking real and imaginary parts in the first equation gives $a + b = -c$, $a = c + d$. Comparing the latter with the second equation above, we obtain $b = 0$ and thus $a = -c$, $d = a - c = -2c$. We already know that the matrix is defined only up to a scalar multiple, so we can set $c = 1$ and get $a = -1$, $b = 0$, $d = -2$. Thus

$$g(z) = \frac{-z}{z-2}.$$

Thus we get

$$f(z) = M_C(g(z)) = \frac{\frac{-z}{z-2} - i}{\frac{-z}{z-2} + i} = \frac{(-1-i)z + 2i}{(-1+i)z - 2i} = i\,\frac{z-1-i}{z-1+i}.$$

Q.5 *Find a Möbius transformation f from the unit disc $\mathbb{D}$ onto the upper half-plane $\mathbb{H}$ that takes 0 to i and (when considered as a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) also takes i to 2. Give an explicit formula for $f(z)$.*

S.5 There are many possible solutions. For example:

Solution 1: Similarly to the previous problem we know that the inverse Cayley map $M_{C^{-1}} = \frac{iz+i}{-z+1}$ takes $\mathbb{D}$ to $\mathbb{H}$ but it takes 0 to i and i to -1. Thus the Möbius transformation we are looking for is a composition $f = g \circ M_{C^{-1}}$ with g being a Möbius transformation that takes $\mathbb{H}$ to itself such that $g(i) = i$ and $g(-1) = 2$. We know that such g is a Möbius transformation with real matrix entries. We need to solve

$$\frac{ai+b}{ci+d} = i, \quad \frac{-a+b}{-c+d} = 2$$

with real a, b, c, d such that $ad - bc \neq 0$. This is equivalent to

$$ai + b = -c + di, \quad -a + b = -2c + 2d.$$

Taking real and imaginary parts in the first equation gives $a = d$ and $b = -c$. If we plug this into the second equation above, we get $-a + b = -c - d = -2c + 2d$, hence $c = 3d$. We already know that the matrix is defined only up to a scalar multiple, so we can set $d = 1$ and get $a = d = 1$, $c = 3$, $b = -3$. Thus

$$g(z) = \frac{z-3}{3z+1}.$$

Thus we get

$$f(z) = g(M_{C^{-1}}(z)) = \frac{\frac{iz+i}{-z+1} - 3}{3\frac{iz+i}{-z+1} + 1} = \frac{(3+i)z - 3 + i}{(-1+3i)z + 1 + 3i} = i\,\frac{(1-3i)z + 1 + 3i}{(-1+3i)z + 1 + 3i}.$$

Solution 2: Instead we can set $f = M_{C^{-1}} \circ g$ where g is a Möbius transformation that takes $\mathbb{D}$ to itself and satisfies $g(0) = 0$ and

$$g(i) = M_C(2) = \frac{2-i}{2+i} = \frac{3-4i}{5} = e^{-i\arctan(4/3)}$$

where we take the branch $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$. From the lectures we know that such a function g is given by $g(z) = e^{i\theta} \frac{z-z_0}{\overline{z_0}z-1}$ where $\theta \in \mathbb{R}$ and $z_0 \in \mathbb{D}$ is the unique point such that $g(z_0) = 0$. Here $z_0 = 0$, hence $g(z) = -e^{i\theta} z$. This implies

$$g(z) = \frac{g(i)}{i} z = -ig(i)z.$$

If one works with $g(i) = \frac{2-i}{2+i}$, then one gets $g(z) = \frac{1+2i}{1-2i} iz$ and hence

$$f(z) = M_{C^{-1}}(g(z)) = \frac{(-1-2i)z + 2 + i}{(2-i)z + 1 + 2i}.$$

If one works with $g(i) = \frac{3-4i}{5}$ (which is the same as before, just written differently), then one gets $g(z) = -\frac{4+3i}{5} z$ and hence

$$f(z) = M_{C^{-1}}(g(z)) = \frac{(3-4i)z + 5i}{(4+3i)z + 5}.$$

Finally, if one works with $g(i) = e^{-i \arctan(4/3)}$ (again the same), then

$$g(z) = e^{-i(\arctan(4/3)+\pi/2)} z = -e^{i \arctan(3/4)}$$

and then insert this into $f(z) = M_{C^{-1}}(g(z))$.

Q.6 *Let $\Omega \subset \mathbb{C}$ be the set of all complex numbers z for which Re$(z) > -1$ and Im$(z) > -1$.*
(a) Find a biholomorphic map f from Ω onto the upper half-plane. Give an explicit formula for $f(z)$.
(b) Hence, find a biholomorphic map $\tilde{f}$ from Ω onto the open unit disc. Give an explicit formula for $\tilde{f}(z)$.

S.6 (a) Consider the mappings f_1 and f_2 given by

$$f_1(z) := z + 1 + i \qquad \text{(translation by } 1+i\text{)},$$
$$f_2(z) := z^2.$$

These mappings are biholomorphic on the domains in question. The first takes Ω to the (open) 1st quadrant, the second takes this quadrant to the upper half-plane. So the composite $f = f_2 \circ f_1$ is biholomorphic and maps Ω onto the upper half plane. We have

$$f(z) \;=\; f_2(f_1(z)) \;=\; (z+1+i)^2.$$

(b) We know a biholomorphic map from the upper half-plane to the unit disc; namely the Cayley Map $f_3 := M_C(z) = \frac{z-i}{z+i}$. Thus,

$$\tilde{f}(z) := \; (f_3 \circ f)(z) = M_C\left((z+1+i)^2\right) \;=\; \frac{(z+1+i)^2 - i}{(z+1+i)^2 + i} \;=\; \frac{z^2 + 2(1+i)z + i}{z^2 + 2(1+i)z + 3i}$$

is the map required.

Q.7 *Does there exist a biholomorphic map taking the closed upper half of the unit disc onto the closed unit disc; i.e. from $\{z \in \mathbb{C} : |z| \leq 1,\ \text{Im}\, z \geq 0\}$ onto $\{w \in \mathbb{C} : |w| \leq 1\}$? Either find an example of such a map or show that none exists. Hint: The boundary would be mapped to the boundary.*

S.7 The boundary of the upper-half disc contains the two sections; a straight line $\{z \in \mathbb{R} : -1 \leq z \leq 1\}$ from -1 to 1 and a circular arc $\{z \in \mathbb{C} : |z| = 1,\ \text{Im}\, z \geq 0\}$ from -1 to 1. These meet at right angles at the points -1 and 1. Their images under a conformal map would also meet at right angles, so the union of their images cannot be the unit circle. As any biholomorphic map is conformal no such map exists.

Q.8 *Use standard examples to find a biholomorphic map from the upper half $\Omega := \{z \in \mathbb{D} : \text{Im}(z) > 0\}$ of the unit disc onto the unit disc $\mathbb{D}$.*

S.8 We recall that the Cayley map

$$f(z) := M_C(z) = \frac{z-i}{z+i}$$

takes the upper right quadrant to the lower part of the unit disc. Consequently its inverse takes the lower part of the unit circle to the upper right quadrant. This implies that

$$f(z) = (M_C^{-1}(z))^2 = \left(\frac{iz+i}{-z+1}\right)^2$$

takes the lower half of the unit disc onto the upper half-plane $\mathbb{H}$. We also know that the Cayley Map $M_C(w) = \frac{w-i}{w+i}$ takes us from $\mathbb{H}$ to the unit disc (and takes i to 0). Hence, if we compose these two maps, we will obtain a biholomorphic map from the lower half of the unit disc onto the unit disc. Thus, to obtain a map from the upper half of the unit disc to the unit disc we need only rotate first by π degrees (so multiply by $e^{i\pi} = -1$). Thus $\tilde{f} := (M_{C^{-1}} \circ f)\left(e^{i\pi}z\right) = (M_{C^{-1}} \circ f)(-z)$ is the required map.

Explicitly, we have

$$\tilde{f}(z) = (M_C \circ f)(-z) = \frac{\frac{(i(-z)+i)^2}{(-(-z)+1)^2} - i}{\frac{(i(-z)+i)^2}{(-(-z)+1)^2} + i} = \frac{(i-iz)^2 - i(z+1)^2}{(i-iz)^2 + i(z+1)^2} = \frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2}.$$

Q.9 *Consider the map $z \to z^2$.*
(i) Find and sketch the images of the lines Im $z = b$ (for $0 < b < 1$).
Hint: find a parametrisation for the lines.
(ii) Find the image of $\{z : 0 < \text{Im}\, z < 1\}$ under this map.

S.9 (i) The image of a point $z = t + ib$ on the line $\text{Im}(z) = b$ is $f(t+ib) = (t+ib)^2 = t^2 - b^2 + 2ibt$. These points form a curve parametrised by $(t^2 - b^2, 2bt)$ in the complex plane and has equation $x = \frac{y^2}{4b^2} - b^2$. As $t \in \mathbb{R}$ this forms the entire parabola.
(ii) The set $\{z : 0 < \text{Im}\, z < 1\}$ is the union of the lines $\text{Im}\, z = b$ for $0 < b < 1$. We notice that the resulting parabolas do not intersect. Indeed if

$$\frac{y^2}{4b_1^2} - b_1^2 = \frac{y^2}{4b_2^2} - b_2^2$$

for some $b_1 \neq b_2$ with $b_1, b_2 \in (0,1)$ then

$$\frac{y^2}{4}\left(\frac{1}{b_1^2} - \frac{1}{b_2^2}\right) = b_1^2 - b_2^2$$

which implies that

$$\frac{y^2}{4b_1^2 b_2^2} = -1.$$

Following the same calculation we actually notice that if $b_1 < b_2$ then

$$\frac{y^2}{4b_1^2} - b_1^2 > \frac{y^2}{4b_2^2} - b_2^2$$

for all y. We conclude that the image of $\{z : 0 < \text{Im}\, z < 1\}$ is the domain bounded by the "boundary parabolas" which correspond to $\text{Im}\,(z) = 0$ and $\text{Im}\,(z) = 1$.
We have seen that $\text{Im}\,(z) = 1$ goes to the parabola $x = \frac{y^2}{4} - 1$. The case of $\text{Im}\,(z) = 0$ needs to be looked at more carefully but following the same idea: the image of a point $z = t$ on the real axis is $f(t) = t^2$, which lies on the line segment $\mathbb{R}_{\geq 0} = [0, \infty)$. We conclude that the image of $\{z : 0 < \text{Im}\, z < 1\}$ is the inside of the parabola $x = \frac{y^2}{4} - 1$ with the positive real axis $\mathbb{R}_{\geq 0}$ removed.

Q.10 *Describe the image of*
(i) $\{z : |z-1| > 1\}$ under $z \to w = \frac{z}{z-2}$
(ii) $\{z : |z-i| < 1, \text{Re}\, z < 0\}$ under $z \to w = \frac{z-2i}{z}$

S.10 (i) Note that the set in question is the exterior of the circle centred at $z = 1$ of radius 1. Clearly, $0 \mapsto 0$, $2 \mapsto \infty$, $1 + i \mapsto \frac{1+i}{-1+i} = -i$. Hence the circle $|z - 1| = 1$ is mapped by the given Möbius transformation to the imaginary axis. Note that $-1 \mapsto 1/3$, which is in the half-plane $\mathbb{H}_R = \{z : \text{Re}(z) > 0\}$ *[or that $1 \mapsto -1/2$, which is in the half-plane $\mathbb{H}_L = \{z : Re(z) < 0\}$]* so the exterior of of the circle $|z - 1| = 1$ maps to the right half-plane $\mathbb{H}_R$.

(ii) We are interested in the region enclosed by the left half of circle $|z - i| = 1$ and the imaginary axis. We have $0 \mapsto \infty, 2i \mapsto 0$ and $i \mapsto -1$, so the section of the imaginary axis between 0 and i maps to the negative real axis. We can now either check that, say, $-1 + i \mapsto i$, or just use angle preservation, to conclude that the circular arc from 0 to i (through $-1 + i$) maps to the positive imaginary axis. As $-\frac{1}{2} + i \mapsto \frac{-3+4i}{5}$ we conclude that the image of our domain is the (open) 2nd quadrant.

Q.11 *Construct a biholomorphic map f from $\mathcal{R}$ onto $\mathcal{R}'$, where $\mathcal{R} = \{z : Im\, z < \frac{1}{2}\}$ and $\mathcal{R}' = \{z : |z-1| < 2\}$. Give an explicit formula for $f(z)$.*

S.11 We can do this using a sequence of standard moves. The answer is not unique. Firstly, the translation $g : z \mapsto z - \frac{i}{2}$ takes the given region to the lower half-plane. The rotation $r : z \mapsto e^{i\pi} = -z$ takes the lower half-plane to the upper half-plane. The Cayley Map $M_C : z \mapsto \frac{z-i}{z+i}$ takes the upper half-plane to the unit disc $B_1(0)$. The dilation $s : z \mapsto 2z$ takes $B_1(0)$ to $B_2(0)$. Finally, The translation $h : z \mapsto z + 1$ takes $B_2(0)$ to the disc $B_2(1)$. Putting all these transformations together we have $f := h \circ s \circ M_C \circ r \circ g$ satisfies $f : \mathcal{R} \xrightarrow{\sim} \mathcal{R}'$.

Explicitly,

$$\begin{aligned} f(z) = (h \circ s \circ M_C)\left(r\left(z - \frac{i}{2}\right)\right) = (h \circ s \circ M_C)\left(-z + \frac{i}{2}\right) &= (h \circ s)\left(\frac{(-z + \frac{i}{2}) - i}{(-z + \frac{i}{2}) + i}\right) \\ &= h\left(2\frac{-2z - i}{-2z + 3i}\right) \\ &= 2\frac{-2z - i}{-2z + 3i} + 1 \\ &= \frac{4z + 2i}{2z - 3i} + \frac{2z - 3i}{2z - 3i} \quad = \frac{6z - i}{2z - 3i}. \end{aligned}$$

Q.12 (*a*) *Find the unique Möbius transformation $f(z)$ taking the ordered set of points $\{0, -1, -i\}$ to the ordered set of points $\{1, \infty, i\}$ in $\hat{\mathbb{C}}$.*

(*b*) *Let C_1 be the circle through 0, -1 and i, and let C_2 be the circle through 0, -1 and $-i$. Let $\mathcal{R}$ be the intersection of the interiors of the two circles. Find the image of $\mathcal{R}$ under your map f, and hence construct a biholomorphic map from $\mathcal{R}$ to the set $\Omega := \{w \in \mathbb{C} : -\pi/4 < Arg(w) < \pi/4\}$.*

(*c*) *Find a biholomorphic map from $\mathcal{R}$ to the upper half-plane $\mathbb{H}$.*

S.12 (a) We know that a Möbius transformation $w = f(z)$ preserves the cross-ratio; i.e.,

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)},$$

where here $\{z_1, z_2, z_3\} = \{0, -1, -i\}$ and $\{w_1, w_2, w_3\} = \{1, \infty, i\}$. We solve for w in terms of z. The above simplifies to

$$1 \cdot \left(\frac{1 - i}{w - i}\right) = \frac{(z - (-1))(0 - (-i))}{(z - (-i))(0 - (-1)} = \frac{iz + i}{z + i},$$

so

$$w(iz + i) = (1 - i)(z + i) + i(iz + i) = -iz + i$$

and $f(z) = \frac{-z+1}{z+1}$ is the required Möbius transformation.

(b) Since z_1, z_2 and z_3 above all lie on C_2, the image of C_2 is the straight line through 1 and i. On the other hand, $f(i) = -i$, so the image of C_1 is the line through 1 and $-i$. these lines intersect at the point

$w = 1$ on the real axis. Furthermore, the point $z = -1/2$ lies in the intersection $\mathcal{R}$ of the two interiors and $f(-1/2) = 3$, so f maps $\mathcal{R}$ to the region $\{w \in \mathbb{C} : -\operatorname{Re}(w) + 1 < \operatorname{Im}(w) < \operatorname{Re}(w) - 1\}$ to the right of these two lines. The translation $g(w) = w - 1$ trivially maps this image to Ω and so the function

$$h := g \circ f : \mathcal{R} \xrightarrow{\sim} \Omega,$$

maps $\mathcal{R}$ to Ω biholomorphically. Explicitly we have

$$h(z) = \frac{-z+1}{z+1} - 1 = \frac{-z+1}{z+1} - \frac{z+1}{z+1} = -\frac{2z}{z+1}.$$

(c) We know a map from the 1st quadrant to the upper half-plane (namely $z \mapsto z^2$) so we first seek a map from Ω to the 1st quadrant. Trivially, the rotation $r(z) = e^{i\pi/4}z = \left(\frac{1+i}{\sqrt{2}}\right)z$ does the trick; an anticlockwise rotation by $\pi/4$ degrees. Finally, we apply the map $z \mapsto z^2$ and we find that

$$\tilde{f}(z) := ((r \circ h)(z))^2 = \left(\left(\frac{1+i}{\sqrt{2}}\right)h(z)\right)^2 = \frac{4iz^2}{(z+1)^2}$$

maps $\mathcal{R}$ biholomorphically to $\mathbb{H}$. *[Note that we could also have applied the map $z \mapsto z^2$ directly to Ω to get the right half-plane and then multiplied by i to rotate anticlockwise $\pi/2$ degrees.]*

Q.13 *Consider the region $P \subset \mathbb{C}$ defined by $P = \left\{z \in \mathbb{D} : -3\pi/4 < Arg\, z < 3\pi/4\right\}$.*
(a) Draw P in the complex plane.
(b) Find a biholomorphic map from P onto the upper half-plane $\mathbb{H}$.
(c) Find a biholomorphic map from P onto the lower half-plane $\mathbb{H}_L := \{w \in \mathbb{C} : Im(z) < 0\}$.

S.13 (b) Consider the mappings f_i (for $i = 1..4$) where

$f_1(z) = z^{2/3}$ (defined using principal branch of Log
$f_2(z) = e^{-i\pi/2}z = -iz$ (clockwise rotation through $\pi/2$)
$f_3(z) = M_{C^{-1}}(z) = \frac{iz+i}{-z+1}$ (inverse Cayley map)
$f_4(z) = z^2$

These mappings are all biholomorphic on the regions in question. Thus, the composite function given by $f := f_4 \circ f_3 \circ f_2 \circ f_1$ is a biholomorphic map from of P onto the upper half plane. We have

$$f(z) = (f_4 \circ f_3 \circ f_2)(z^{2/3}) = (f_4 \circ f_3)(-iz^{2/3}) = \left(\frac{i(-iz^{2/3}) + i}{-(-iz^{2/3}) + 1}\right)^2 = \left(\frac{z^{2/3} + i}{iz^{2/3} + 1}\right)^2.$$

[Note that this solution is not unique!]

(c) Compose f above with the reflection (rotation by π degrees) $f_5(z) = e^{i\pi}z = -z$.