1 For every $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{x^n}$ for $x \in [1,\infty)$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $[1,\infty)$, and determine whether convergence is uniform on $[1,\infty)$. For a fixed $R > 1$, determine whether convergence is uniform on $[R,\infty)$.

2 For every $n \in \mathbb{N}$, let $f_n(x) = \arctan(nx)$ for $x \in \mathbb{R}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$ to

$$f(x) = \begin{cases} \pi/2, & \text{if } x > 0. \\ 0, & \text{if } x = 0. \\ -\pi/2, & \text{if } x < 0. \end{cases}$$

Is the convergence uniform?

3 (i) Show that for any $\rho > 0$ the sequence $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converges uniformly on $\{\, z \in \mathbb{C} : \ |z| \geq \rho \,\}$.

(ii) Does $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converge uniformly on $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$?

4 For any $\rho > 0$, show that $\left\{\frac{n}{1+nz}\right\}_{n\in\mathbb{N}}$ converges uniformly on $\{\, z \in \mathbb{C} : \ |z| > \rho \,\}$. Does it converge uniformly on $\mathbb{C}^*$?

5 (i) Show that if $0 < \rho < 1$, then the sequence $\left\{\frac{1}{1+z^n}\right\}_{n\in\mathbb{N}}$ converges uniformly on $\{\, z \in \mathbb{C} : \ |z| \leq \rho \,\}$ to the constant function $f(z) = 1$. On the other hand, show that the sequence converges uniformly on $\{\, z \in \mathbb{C} : \ |z| \geq \rho^{-1} \,\}$ to the constant function $f(z) = 0$.

(ii) Show that the sequence $\left\{\frac{1}{1+z^n}\right\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathbb{D} = \{\, z \in \mathbb{C} : \ |z| < 1 \,\}$.

6 For every $n \in \mathbb{N}$, let $f_n(z) = \sin(z/n)$ for $z \in \mathbb{C}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{C}$. Let ρ be a positive real number. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\{z : |z| \leq \rho\}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathbb{C}$.

7 For every $n \in \mathbb{N}$, let $f_n(x) = \cos\left(1 + \frac{x}{n}\right)$ for $x \in \mathbb{R}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise and determine whether convergence is uniform on $\mathbb{R}$. For fixed $R > 0$, is the convergence uniform on $[0, R]$?

8 Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2}$ converges uniformly on $\left\{z \in \mathbb{C} : \ |z| \leq \frac{1}{\sqrt{2}}\right\}$, and deduce that the limit function is continuous on this set.

9 Prove that $\sum_{n=0}^{\infty} e^{nz}$ converges uniformly on $\{z \in \mathbb{C} : \ \mathrm{Re}(z) \leq -1\}$, but not on $\{z \in \mathbb{C} : \ \mathrm{Re}(z) \leq 0\}$.

10 Let R satisfy $0 < R < 1$. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$ converges uniformly on $\{z \in \mathbb{C} : \ |z| < R\}$. Conclude that the infinite series defines a continuous function on the unit disc $\mathbb{D}$.

11 Prove that each of the following series converge uniformly on the corresponding subset of $\mathbb{C}$:

$$(a) \ \sum_{n=1}^{\infty} \frac{1}{n^2 z^{2n}}, \qquad \text{on} \quad \{\, z \in \mathbb{C} : \ |z| \geq 1 \,\}.$$

$$(b) \ \sum_{n=1}^{\infty} \sqrt{n}\, e^{-nz}, \qquad \text{on} \quad \{\, z \in \mathbb{C} : \ 0 < r \leq \mathrm{Re}(z) \,\}.$$

$$(c) \ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \qquad \text{on} \quad \left\{ z \in \mathbb{C} : \ |z| \leq r < \frac{1}{2} \right\}.$$

$$(d) \ \sum_{n=1}^{\infty} 2^{-n} \cos(nz), \qquad \text{on} \quad \{\, z \in \mathbb{C} : \ |\mathrm{Im}(z)| \leq r < \ln 2 \,\}.$$

12 Given $0 < r < R < \infty$, show that $\displaystyle\sum_{n=1}^{\infty} \frac{\left(z + \frac{1}{z}\right)^n}{n!}$ converges uniformly on $r < |z| < R$. Conclude that the infinite series defines a continuous function on $\mathbb{C}^*$.

13 Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}$ converges uniformly on $|z| < r$, for any $r < 1$. Prove it also converges uniformly on $|z| \geq R$, for any $R > 1$. Conclude that the infinite series defines a continuous function inside and outside the unit circle. What is the situation on the unit circle?