Q.1 *For every $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{x^n}$ for $x \in [1, \infty)$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $[1, \infty)$, and determine whether convergence is uniform on $[1, \infty)$. For a fixed $R > 1$, determine whether convergence is uniform on $[R, \infty)$.*

S.1 Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Clearly $f_n \to f$ pointwise on $[1, \infty)$, because $f_n(1) = 1$ for all n, and if $x > 1$ then $\frac{1}{x^n} \to 0$.

[In full epsilonic detail, for every $\epsilon \in (0, 1)$ and $x > 1$ we may pick any $N > |\ln(\epsilon)|/\ln(x)$, because then, for $n > N$, we have $|f_n(x) - f(x)| = 1/x^n < 1/x^N < \epsilon$.]

Method 1 (using a big theorem): Each f_n is continuous on $[1, \infty)$ and f is not, so by the Uniform Limit Theorem the convergence cannot be uniform on $[1, \infty)$.

Method 2 (by lesser means): Pick any real number $c > 0$ and consider the sequence $x_n = (1 + c)^{1/n}$ in $[1, \infty)$. We have $|f_n(x_n) - f(x_n)| = |1/(1 + c) - 0| = 1/(1 + c)$, so by the test for non-uniform convergence convergence is not uniform.

For $[R, \infty)$, note that for any $x \in [R, \infty)$ we have

$$|f_n(x) - f(x)| = \frac{1}{x^n} \leq \frac{1}{R^n} \to 0 \text{ as } n \to \infty.$$

So, according to a lemma from class the convergence is uniform in $[R, \infty)$.

[If you wish you can prove all of the above directly from the definitions without appealing to any of the Theorems/Lemmas.]

Q.2 *For every $n \in \mathbb{N}$, let $f_n(x) = \arctan(nx)$ for $x \in \mathbb{R}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$ to*

$$f(x) = \begin{cases} \pi/2, & \textit{if } x > 0. \\ 0, & \textit{if } x = 0. \\ -\pi/2, & \textit{if } x < 0. \end{cases}$$

Is the convergence uniform?

S.2 In full epsilonic detail: Let $\epsilon > 0$ and pick $x \in \mathbb{R}$. If $x = 0$ we have $f_n(x) = \arctan(0) = 0$, and so convergence to 0 is trivial. If $x > 1$, then pick any $N > \tan(\pi/2 - \epsilon)/x$. Then for $n > N$ we have

$$|f_n(x) - f(x)| = \pi/2 - \arctan(nx) < \pi/2 - \arctan(x \tan(\pi/2 - \epsilon)/x) = \pi/2 - (\pi/2 - \epsilon) = \epsilon,$$

since arctan is an increasing function. Similarly, if $x < 0$ pick any $N > \tan(\epsilon - \pi/2)/x$ (which is positive). Then for $n > N$ we have

$$|f_n(x) - f(x)| = \arctan(nx) + \pi/2 < \arctan(x \tan(\epsilon - \pi/2)/x) + \pi/2 = (\epsilon - \pi/2) + \pi/2 = \epsilon.$$

Note that each N depends on x so we do not expect uniform convergence. Indeed:

Method 1 (using a big theorem): Each f_n is continuous on $\mathbb{R}$ and f is not, so by the Uniform Limit Theorem convergence cannot be uniform on $\mathbb{R}$.

Method 2 (by lesser means): Pick any real number $0 < c < \pi/2$ and consider the sequence $x_n = \tan(c)/n$ in $\mathbb{R}$. We have $|f_n(x_n) - f(x_n)| = |c - \pi/2| = \pi/2 - c$, so by the test for non-uniform convergence the convergence is not uniform.

Q.3 *(i) Show that for any $\rho > 0$ the sequence $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converges uniformly on $\{ z \in \mathbb{C} : |z| \geq \rho \}$.*

(ii) Does $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converge uniformly on $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$?

S.3 (i) For every z in the set we have $|1/(nz)| = |z|^{-1}(1/n) \to 0$ as $n \to \infty$, so the pointwise limit is the constant function $f(z) = 0$.
For each fixed n we see that $\left|\frac{1}{nz}\right| \leq \frac{1}{\rho n}$ for all $|z| \geq \rho$. Since $\left\{\frac{1}{\rho n}\right\} \to 0$ as $n \to \infty$, it follows from a lemma from class that the given sequence converges uniformly to the function $f(z) = 0$ on the set $\{\, z \in \mathbb{C} : \ |z| \geq \rho \,\}$.

(ii) The pointwise limit on $\mathbb{C}^*$ is the constant function $f(z) = 0$, so, if convergence is uniform then the uniform limit must be $f(z) = 0$. However, for any $c > 0$ consider the sequence $z_n = c/n$ in $\mathbb{C}^*$. We have $|f_n(z_n) - f(z_n)| = 1/c$, so by the test for non-(uniform convergence) the convergence is not uniform.
[Alternatively, by hand: for any fixed n, as $z \to 0$ we see that $\left|\frac{1}{nz}\right|$ tends to infinity. So given $\epsilon > 0$ and any $n \in \mathbb{N}$, we can always find a point $z \in \mathbb{C}$ such that $\left|\frac{1}{nz}\right| \geq \epsilon$; namely $z = 1/(n\epsilon)$ will do. Hence convergence is not uniform.]

Q.4 *For any $\rho > 0$, show that $\left\{\frac{n}{1+nz}\right\}_{n\in\mathbb{N}}$ converges uniformly on $\{\, z \in \mathbb{C} : \ |z| > \rho \,\}$. Does it converge uniformly on $\mathbb{C}^*$?*

S.4 Given any fixed $z \in \{z \in \mathbb{C} : \ |z| > \rho\}$ we have $\lim_{n\to\infty} \frac{n}{1+nz} = \lim_{n\to\infty} \frac{1}{(1/n)+z} = \frac{1}{z}$, so the pointwise limit on the set is the function $f(z) = 1/z$. Also, for each fixed n, we see that

$$\left|\frac{n}{1+nz} - \frac{1}{z}\right| = \left|\frac{1}{z(1+nz)}\right|.$$

However, for $|z| > \rho$ and n sufficiently large ($n > 1/\rho$, say), we have that $|z(1+nz)| > \rho|1+nz| \geq \rho(\rho n - 1)$, so that

$$|f_n(z) - f(z)| = \left|\frac{n}{1+nz} - \frac{1}{z}\right| \leq \frac{1}{\rho(\rho n - 1)}.$$

Let $\epsilon > 0$. Since $\left\{\frac{1}{\rho(\rho n-1)}\right\} \to 0$ as $n \to \infty$, we can find $N \in \mathbb{N}$ such that $\frac{1}{\rho(\rho n-1)} < \epsilon$ for all $n > N$; this proves that converge is uniform (to the function $f(z) = 1/z$) on $\{z \in \mathbb{C} : \ |z| > \rho\}$.

To see if the convergence is not uniform on all of $\mathbb{C}^*$ we wish to construct a sequence z_n in $\mathbb{C}^*$ and find a constant $c > 0$ such that $|f_n(z_n) - f(z_n)| = c$. Let's just find such a sequence of strictly positive real numbers. Let $c > 0$. Then we want z_n such that $\frac{1}{z_n(1+nz_n)} = c$. By rearanging this equation and completing the square we see that

$$z_n = \frac{1}{2n} + \sqrt{\frac{1}{4n^2} + \frac{1}{cn}}$$

does the trick. Each of these is clearly in $\mathbb{C}^*$, so convergence is not uniform.

[Alternatively: To show convergence is not uniform on all of $\mathbb{C}^*$ we can argue more directly. Given n, we can always find a point $z \in \mathbb{C}^*$ such that $1 + nz = 0$ (namely, $z = -1/n$). Hence, nearby $z = -1/n$, the difference $\left|\frac{1}{z(1+nz)}\right|$ becomes arbitrarily large. That is, for any $\epsilon > 0$ and any $n \in \mathbb{N}$ we can find z such that $|f_n(z) - f(z)| = \left|\frac{1}{z(1+nz)}\right| \geq \epsilon$.]

Q.5 *(i) Show that if $0 < \rho < 1$, then the sequence $\left\{\frac{1}{1+z^n}\right\}_{n\in\mathbb{N}}$ converges uniformly on $\{\, z \in \mathbb{C} : \ |z| \leq \rho \,\}$ to the constant function $f(z) = 1$. On the other hand, show that the sequence converges uniformly on $\{\, z \in \mathbb{C} : \ |z| \geq \rho^{-1} \,\}$ to the constant function $f(z) = 0$.*

(ii) Show that the sequence $\left\{\frac{1}{1+z^n}\right\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathbb{D} = \{\, z \in \mathbb{C} : \ |z| < 1 \,\}$.

S.5 (i) Let $f_n(z) = 1/(1 + z^n)$ be defined on $\{z \in \mathbb{C} : |z| \leq \rho\}$ and let $f(z) = 1$. Then, since by the reverse triangle inequality $|1 + z^n| = |1 - (-z^n)| \geq \big|\, |1| - |(-z^n)| \,\big| = 1 - |z|^n$ for z with $|z| < 1$, we have for z in $\{z \in \mathbb{C} : |z| \leq \rho\}$ that

$$|f_n(z) - f(z)| \quad = \quad \left|\frac{1}{1+z^n} - 1\right| \quad = \quad \left|\frac{z^n}{1+z^n}\right| \quad \leq \quad \frac{|z|^n}{1-|z|^n} \quad \leq \quad \frac{\rho^n}{1-\rho^n} \to 0 \ \text{ as } n \to \infty.$$

Hence $\{f_n\}$ converges uniformly on $\{z \in \mathbb{C} : |z| \leq \rho\}$ to the constant function $f(z) = 1$.

Now consider points in $\{ z \in \mathbb{C} : \ |z| \geq \rho^{-1} \}$. Here $|z| > 1$, so $|1+z^n| = |1-(-z^n)| \geq \big| |1| - |(-z^n)| \big| = |z|^n - 1$. Then

$$|f_n(z) - f(z)| \quad = \quad \left| \frac{1}{1+z^n} \right| \quad \leq \quad \frac{1}{|z|^n - 1} \quad \leq \quad \frac{1}{\rho^{-n} - 1} \quad = \quad \frac{\rho^n}{1-\rho^n} \to 0 \ \text{ as } n \to \infty.$$

Therefore $\{f_n\}$ converges uniformly on $\{ z \in \mathbb{C} : \ |z| \geq \rho^{-1} \}$ to the constant function $f(z) = 0$.

(ii) As above, the sequence converges pointwise on $\{z : |z| < 1\}$ to the constant function $f(z) = 1$. To show convergence is not uniform it is enough to find $c > 0$ and a sequence z_n in $\mathbb{D}$ such that $|f_n(z_n) - f(z_n)| = c$. Let's just find such a sequence z_n in the open unit interval $(0, 1)$ on the real line. If

$$|f_n(z_n) - f(z_n)| = \frac{|(z_n)^n|}{|1 + (z_n)^n|} = \frac{(z_n)^n}{1 + (z_n)^n} = c,$$

then rearranging gives us $z_n = \left(\frac{c}{1-c}\right)^{1/n}$. So, we may for example choose $c = 1/3$, then $z_n = \left(\frac{c}{1-c}\right)^{1/n} = \frac{1}{2^{1/n}} \in \mathbb{D}$ and $|f_n(z_n) - f(z_n)| = 1/3$, so the convergence is not uniform in $\mathbb{D}$.

[Alternatively: Note that for a fixed n, we have

$$\lim_{z\to 1} |f_n(z) - f(z)| = \lim_{z\to 1} \frac{|z^n|}{|1+z^n|} = \frac{1}{1+1} = \frac{1}{2}\,.$$

So let $\epsilon = 1/4$, say. Then for any n we can find $z \in \mathbb{D}$ such that $|f_n(z) - f(z)| \geq \frac{1}{2} - \epsilon = \epsilon$. Hence the convergence is not uniform.]

Q.6 *For every $n \in \mathbb{N}$, let $f_n(z) = \sin(z/n)$ for $z \in \mathbb{C}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{C}$. Let ρ be a positive real number. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\{z : |z| \leq \rho\}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathbb{C}$.*

S.6 For fixed z, $\lim_{n\to\infty} z/n = 0$, so, since $\sin(z)$ is continuous at $z = 0$, it follows that, for fixed z, $\lim_{n\to\infty} \sin(z/n) = \sin(0) = 0$. Hence $\{f_n\}$ converges pointwise on $\mathbb{C}$ to the zero function $f(z) = 0$.

For each fixed n,

$$|\sin(z/n) - f(z)| = |\sin(x/n)\cosh(y/n) + i\cos(x/n)\sinh(y/n)| \leq |\sin(x/n)\cosh(y/n)| + |\sinh(y/n)|,$$

so, for $|z| \leq \rho$,

$$|\sin(z/n) - f(z)| \leq (\rho/n)\cosh(\rho/n) + \sinh(\rho/n).$$

Putting s_n equal to the RHS of the above inequality, we see that $\lim_{n\to\infty} s_n = 0$. Hence, by a lemma from class, we have that $\{f_n\}$ converges uniformly to the zero function on $\{z : |z| \leq \rho\}$.

[Alternatively (without using the Lemma): Note $|\sin(z/n) - f(z)|$ can be made arbitraily small for large n independent of z, that is, for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that for $n > N$ we have $s_n < \epsilon$, so $\{f_n\}$ converges uniformly to the zero function on $\{z : |z| \leq \rho\}$.]

To see if the convergence is not uniform on $\mathbb{C}$ we wish to construct a sequence $\{z_n\}_{n\in\mathbb{N}}$ and find a constant $c > 0$ such that $|f_n(z_n) - f(z_n)| = c$. The sequence $z_n = in$ with $c = \sinh(1) > 0$ does the trick, for then $|f_n(z_n) - f(z_n)| = |\sin(z_n/n)| = |\sin(i)| = \sinh(1)$. Thus, convergence is not uniform.

[Alternatively, just say that for fixed n, we have $lim_{y\to\infty}|\sin(iy/n)| = lim_{y\to\infty}|\sinh(y/n)| = \infty$. Hence for every n, $|f_n(z) - f(z)|$ is unbounded. So convergence isn't uniform on $\mathbb{C}$.]

Q.7 *For every $n \in \mathbb{N}$, let $f_n(x) = \cos\left(1 + \frac{x}{n}\right)$ for $x \in \mathbb{R}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise and determine whether convergence is uniform on $\mathbb{R}$. For fixed $R > 0$, is the convergence uniform on $[0, R]$?*

S.7 For fixed x, $\lim_{n\to\infty} x/n = 0$, so, since $\cos(x)$ is continuous at $x = 1$, it follows that, for fixed x we have $\lim_{n\to\infty}\cos(1 + x/n) = \cos(1)$. Hence $\{f_n\}$ converges pointwise on $\mathbb{R}$ to the constant function $f(x) = \cos(1)$.

To see if the convergence is not uniform on $\mathbb{R}$ we wish to construct a sequence x_n in $\mathbb{R}$ and find a constant $c > 0$ such that $|f_n(x_n) - f(x_n)| = c$. The sequence $x_n = (\frac{\pi}{2} - 1)n$ with $c = \cos(1) > 0$ does the trick, for then $|f_n(x_n) - f(x_n)| = |\cos(1 + x_n/n) - \cos(1)| = |\cos(\pi/2) - \cos(1)| = \cos(1)$. Thus, convergence is not uniform.

[Alternatively: Note that for fixed n, $\cos\left(1 + \frac{x}{n}\right)$ periodically takes value 0 as $x \to \infty$, so for all n and any $\epsilon > 0$, there must be some $x \in \mathbb{R}$ such that $|f_n(x) - \cos(1)| \geq \epsilon$. So convergence is not uniform on $\mathbb{R}$.]

We now consider what happens on $[0, R]$. For fixed n,

$$\left|\cos\left(1 + \frac{x}{n}\right) - \cos(1)\right| \leq \frac{x}{n} \leq \frac{R}{n} \quad \text{(by the Mean Value Theorem from last year).}$$

So, taking $s_n = \frac{R}{n}$, we see that $\{s_n\} \to 0$ as $n \to \infty$. (By Lemma 5.6 part 1.), this shows that we have uniform convergence (to the constant function $f(x) = \cos(1)$ on $[0, R]$.

Q.8 *Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2}$ converges uniformly on $\left\{z \in \mathbb{C} :\ |z| \leq \frac{1}{\sqrt{2}}\right\}$, and deduce that the limit function is continuous on this set.*

S.8 Let $f_k(z) = \frac{2^k z^{2k}}{k^2}$. First note that for $|z| \leq \frac{1}{\sqrt{2}}$ we have $|z|^{2k} \leq 1/2^k$, so

$$|f_k(z)| \quad = \quad \left|\frac{2^k z^{2k}}{k^2}\right| \quad = \quad \frac{2^k |z|^{2k}}{k^2} \quad \leq \quad \frac{1}{k^2}.$$

Now $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent series, so, by the Weierstrass M-test, the series converges uniformly on $|z| \leq \frac{1}{\sqrt{2}}$. Since each term of the series is continuous, we see (since uniform limits of continuous functions are continuous) that the limit function $f(z) := \sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2}$ is continuous. Note that we do not know what $f(z)$ is, just that it exists and is continuous.

Q.9 *Prove that $\sum_{n=0}^{\infty} e^{nz}$ converges uniformly on $\{z \in \mathbb{C} :\ Re(z) \leq -1\}$, but not on $\{z \in \mathbb{C} :\ Re(z) \leq 0\}$.*

S.9 Let $f_n(z) = e^{nz}$ and write $z = x + iy$, with $x \leq -1$. We have

$$|f_n(z)| \;=\; |e^{nz}| \;=\; |e^{nx} e^{iny}| \;=\; e^{nx} |e^{iny}| \;=\; e^{nx} \;\leq\; e^{-n} \;=\; \left(\frac{1}{e}\right)^n.$$

As $\frac{1}{e} < 1$, the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is convergent (it is just a geometric series). It follows by the Weierstrass M-test that $\displaystyle\sum_{n=1}^{\infty} e^{nz}$ converges uniformly on $\{z \in \mathbb{C} :\ \mathrm{Re}(z) \leq -1\}$.

The series is not even pointwise convergent on $\{z \in \mathbb{C} :\ \mathrm{Re}(z) \leq 0\}$; take for example $z = 0$ for then the partial sums $\sum_{n=0}^{N} e^{nz} = N + 1$ clearly do not converge. Thus the series does not converge uniformly on the set $\{z \in \mathbb{C} :\ \mathrm{Re}(z) \leq 0\}$.

Q.10 *Let R satisfy $0 < R < 1$. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1 + z^n}$ converges uniformly on $\{z \in \mathbb{C} :\ |z| < R\}$. Conclude that the infinite series defines a continuous function on the unit disc $\mathbb{D}$.*

S.10 Let $f_n(z) = \frac{z^n}{1+z^n}$. Notice that for z in $\{z \in \mathbb{C} : |z| < R\}$ we have $|z| < 1$. So, by the reverse triangle inequality,

$$|1 + z^n| = |1 - (-z^n)| \geq \big| |1| - |(-z^n)| \big| = \big|1 - |z|^n\big| = 1 - |z|^n \geq 1 - R^n.$$

Thus, for z in $\{z \in \mathbb{C} : |z| < R\}$, we have

$$|f_n(z)| = \left|\frac{z^n}{1+z^n}\right| \leq \frac{|z|^n}{1-R^n} < \frac{R^n}{1-R^n}.$$

Let $M_n = \frac{R^n}{1-R^n}$, then (by the ratio test) the sum $\sum_{n=1}^{\infty} M_n$ converges.

$$\Big[\,\textit{Details: } L = \lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{R^{n+1}/(1-R^{n+1})}{R^n/(1-R^n)} = \lim_{n\to\infty} \frac{R(1-R^n)}{1-R^{n+1}} = R < 1.\,\Big]$$

Thus, by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$ converges uniformly on $\{z \in \mathbb{C} : |z| < R\}$.

To see that the series is continuous on the unit disc, note that for every point $z \in \mathbb{D}$ we can find $0 < R < 1$ such that $z \in B_R(0)$. Since the series converges uniformly on $B_R(0)$ and each function f_n is continuous, the limit function is continuous at z.

[Note that here we have really used locally uniform convergence - we have found an open set $B_R(0)$ containing z, on which the series converges uniformly.]

Q.11 *Prove that each of the following series converge uniformly on the corresponding subset of $\mathbb{C}$:*

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{n^2 z^{2n}}, \qquad \text{on} \quad \{\, z \in \mathbb{C} : |z| \geq 1 \,\}.$$

$$(b)\ \sum_{n=1}^{\infty} \sqrt{n}\, e^{-nz}, \qquad \text{on} \quad \{\, z \in \mathbb{C} : 0 < r \leq Re(z) \,\}.$$

$$(c)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \qquad \text{on} \quad \left\{\, z \in \mathbb{C} : |z| \leq r < \frac{1}{2} \right\}.$$

$$(d)\ \sum_{n=1}^{\infty} 2^{-n} \cos(nz), \qquad \text{on} \quad \{\, z \in \mathbb{C} : |Im(z)| \leq r < \ln 2 \,\}.$$

S.11 As in the previous solutions, simply compare each series to

$$(a)\ \sum n^{-2}; \quad (b)\ \sum \sqrt{n} e^{-nr}; \quad (c)\ \sum \frac{(2r)^n}{1 - r^{2n}}; \quad (d)\ \sum 2^{-n} e^{nr};$$

respectively. All converge (we know the series in (a) converges from Analysis I, the series in (b), (c), (d) converge via the ratio test) so the Weierstrass M-test implies uniform convergence of each series.

Q.12 *Given $0 < r < R < \infty$, show that $\sum_{n=1}^{\infty} \frac{\left(z + \frac{1}{z}\right)^n}{n!}$ converges uniformly on $r < |z| < R$. Conclude that the infinite series defines a continuous function on $\mathbb{C}^*$.*

S.12 When $r < |z| < R$ we have

$$\left|z + \frac{1}{z}\right| \leq |z| + \left|\frac{1}{z}\right| \leq R + \frac{1}{r}.$$

Let $M_n = \frac{\left(R + \frac{1}{r}\right)^n}{n!}$. Then, the sum $\sum_{n=1}^{\infty} M_n$ converges (by the ratio test).

$$\Big[\,\textit{Details: } L = \lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{\left(R + \frac{1}{r}\right)^{n+1}/(n+1)!}{\left(R + \frac{1}{r}\right)^n/n!} = \lim_{n\to\infty} \frac{R + \frac{1}{r}}{n+1} = 0 < 1.\,\Big]$$

Thus, the series $\sum_{n=1}^{\infty} \frac{\left(z+\frac{1}{z}\right)^n}{n!}$ converges uniformly on $\{z \in \mathbb{C} : r < |z| < R\}$ by the Weierstrass M-test.

To see that the series is continuous on $\mathbb{C}^*$, note that for every point $z_0 \in \mathbb{C}^*$ we can find $0 < r < R < \infty$ such that $r < |z_0| < R$. Since the series converges uniformly on $\{z \in \mathbb{C} : r < |z| < R\}$, this shows that the series converges locally uniformly on $\mathbb{C}^*$. Thus by a theorem from class, since all the terms of the series are continuous on $\mathbb{C}^*$, the limit function is continuous.

Q.13 *Prove that* $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ *converges uniformly on* $|z| < r$*, for any* $r < 1$*. Prove it also converges uniformly on* $|z| \geq R$*, for any* $R > 1$*. Conclude that the infinite series defines a continuous function inside and outside the unit circle. What is the situation on the unit circle?*

S.13 If $|z| < r$ for some $r < 1$, then $|1 + z^{2n}| \geq 1 - r^{2n}$. Therefore, for $|z| < r$

$$\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{r^n}{1-r^{2n}}.$$

The series $\sum \frac{r^n}{1-r^{2n}}$ is convergent (by the ratio test). So, by the Weierstrass M-test, the series is uniformly convergent on the set $\{z \in \mathbb{C} : \ |z| \leq r\}$, for any $r < 1$.

If $|z| \geq R$ (for $R > 1$), then

$$\left|\frac{z^n}{1+z^{2n}}\right| = \left|\frac{1}{z^n(1/z^{2n}+1)}\right| \leq \frac{1}{R^n}\frac{1}{1-1/R^{2n}} = \frac{R^n}{R^{2n}-1}.$$

You can use now the same arguments as above to conclude that the series converges uniformly on $|z| \geq R$, for any $R > 1$.

To see that the series is continuous inside the unit disc, note that for every point $z_0 \in \mathbb{D}$ we can find $0 < r < 1$ such that $|z_0| < r$, and so $z_0 \in B_r(0)$. Since the series converges uniformly on the open ball $B_r(0)$ and each function f_n is continuous, the limit function is continuous at z.

[Note that here we have really used <u>*locally*</u> *uniform convergence - we have found an open set* $B_r(0)$ *containing* z_0*, on which the series converges uniformly.]*

Similarly, for $|z_0| > 1$ there is an $R > 1$ such that $|z_0| > R$ so that z_0 is in the open set $\{z \in \mathbb{C} : |z| > R\}$, on which the series converges uniformly. Thus the series is continuous on $\{z \in \mathbb{C} : |z| > 1\}$.

On the unit circle the series does not even converge pointwise. For example, take $z = 1$.