Q.1 *Find parametrisations for the following curves:*
*(a) The line segment from $-1$ to $1-i$,*
*(b) The circle of radius 2 centred at $1+i$,*
*(c) The ellipse $x^2+4y^2=1$.*

S.1 (a) The line segment through $z_0$ and $z_1$ is obtained by starting at $z_0$ and going towards $z_1$. This is parametrised by $\gamma(t)=z_0+t(z_1-z_0),\ 0\le t\le 1$. Thus, in this case a possible parametrisation is $\gamma(t)=-1+2t-it,\ 0\le t\le 1$.
(b) A suitable parametrisation is $\gamma(t)=1+i+2e^{2\pi it},\ 0\le t\le 1$.
(c) Here we could take $\gamma(t)=\cos t+(i/2)\sin t, 0\le t\le 2\pi$.

Q.2 *Let $\gamma:[a,b]\to\mathbb{C}$ be a $C^1$-curve, and define $(-\gamma):[-b,-a]\to\mathbb{C}$, by $(-\gamma)(t):=\gamma(-t)$. Show that for any $f$ such that $\int_\gamma f(z)\,dz$ is well defined we have that*

$$\int_{-\gamma} f(z)\,dz=-\int_\gamma f(z)\,dz$$

S.2 We have $(-\gamma)'(t)=(\gamma(-t))'=-\gamma'(-t)$ and hence

$$\int_{-\gamma} f(z)dz=\int_{-b}^{-a} f(\gamma(-t))(\gamma(-t))'\,dt=-\int_{-b}^{-a} f(\gamma(-t))\gamma'(-t)\,dt$$

We make the change of variables $s:=-t$, that is $ds=-dt$ and then the last integral is equal to

$$\int_b^a f(\gamma(s))\gamma'(s)\,ds=-\int_a^b f(\gamma(s))\gamma'(s)\,ds=-\int_\gamma f(z)\,dz.$$

Q.3 *Let $\gamma:[0,4]\to\mathbb{C}$ be the curve given by*

$$\gamma(t)=\begin{cases} t, & 0\le t\le 1,\\ 1+i(t-1), & 1\le t\le 2,\\ 3-t+i, & 2\le t\le 3,\\ i(4-t), & 3\le t\le 4.\end{cases}$$

*Draw this curve in the complex plane and directly compute $\int_\gamma e^z dz$ (without using the Fundamental Theorem of Calculus).*

S.3 The curve is the square (of side length 1) with vertices at $0, 1, 1+i$ and $i$. Do the integrals along each side separately and add the answers. So, integrating along the side that lies on the real axis we get

$$\int_0^1 e^t dt=e-1.$$

Along the next side we get

$$\int_1^2 e^{1+i(t-1)}i dt=e^{1+i(t-1)}|_1^2=e^{1+i}-e.$$

The other two sides of the square give

$$\int_2^3 e^{3-t+i}(-1)dt=e^{3-t+i}|_2^3=e^i-e^{1+i},$$

and

$$\int_3^4 e^{i(4-t)}(-i)dt=e^{i(4-t)}|_3^4=1-e^i.$$

So the total integral is 0 - which it had to be by FTC since $e^z$ has an antiderivative and the contour is closed!

Q.4 *Calculate* $\int_\gamma |z|\,dz$ *when* $\gamma$ *is the straight line from* $-i$ *to* $i$*, and when* $\gamma$ *is the segment of the unit circle which joins* $-i$ *to* $i$ *in the right hand half-plane.*

S.4 In the first case, write $\gamma(t) = it$ for $-1 \le t \le 1$, so $\gamma'(t) = i$. We need to break the path integral into two pieces so $|\gamma(t)| = -t$ for $-1 \le t \le 0$ and $|\gamma(t)| = t$ for $0 \le t \le 1$:

$$\begin{aligned}\int_\gamma |z|\,dz &= \int_{t=-1}^{0} -t\,i\,dt + \int_{t=0}^{1} t\,i\,dt \\ &= -i[t^2/2]_{t=-1}^{0} + i[t^2/2]_{t=0}^{1} \\ &= i/2 + i/2 = i.\end{aligned}$$

In the second case, we write $\gamma(t) = e^{it}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, so $\gamma'(t) = ie^{it}$ and $|\gamma(t)| = 1$.

$$\begin{aligned}\int_\gamma |z|\,dz &= \int_{t=-\pi/2}^{\pi/2} ie^{it}\,dt \\ &= [e^{it}]_{t=-\pi/2}^{\pi/2} \\ &= i - (-i) = 2i.\end{aligned}$$

Note that, although the curves have the same endpoints, the integral of the curves is different.

Q.5 *Calculate* $\int_\gamma \frac{1}{z}\,dz$*, where* $\gamma(t) = (1+2t)e^{4\pi it}$ *for* $0 \le t \le 1$.

S.5 We have

$$\gamma'(t) = 2e^{4\pi it} + 4\pi i(1+2t)e^{4\pi it}.$$

Therefore

$$\begin{aligned}\int_\gamma \frac{1}{z}\,dz &= \int_{t=0}^{1} \frac{2e^{4\pi it} + 4\pi i(1+2t)e^{4\pi it}}{(1+2t)e^{4\pi it}}\,dt \\ &= \int_{t=0}^{1} \left(\frac{2}{1+2t} + 4\pi i\right)\,dt \\ &= \Big[\log(1+2t) + 4\pi it\Big]_{t=0}^{1} \\ &= \log(3) + 4\pi i.\end{aligned}$$

Note that when evaluating the integral, we used the fact that on the interval $0 \le t \le 1$ there is a well defined branch of $\log(1+2t)$. Indeed, since $1+2t$ for $0 \le t \le 1$ is real, this is just the usual real valued logarithm (often denoted by $\ln$).

Comment: $\gamma$ is a spiral that winds around 0 twice, with linearly increasing radius. A candidate for the antiderivative of $1/z$ is $\log z$ but this doesn't have a holomorphic branch along the whole of $\gamma$ (problem: branch cut). So we can't use the Fundamental Theorem of Calculus.

Q.6 *Let* $\gamma_\rho$ *be the curve* $\gamma_\rho(\theta) := \rho e^{i\theta}$ *with* $0 \le \theta \le \pi$*. Let* $z^{\frac{1}{2}}$ *be the branch of square root corresponding to the branch of* $\log$ *with argument in* $(-\pi/2, 3\pi/2]$*, that is, if* $z = \rho e^{i\theta}$ *with* $\theta \in (-\pi/2, 3\pi/2]$ *then* $z^{\frac{1}{2}} = \sqrt{\rho}e^{i\theta/2}$*. Show that*

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} \frac{z^{1/2}}{z^2+1} = 0.$$

S.6 Our strategy is to use the Estimation Lemma. We note that for any $z$ with $|z| = \rho$ we have that

$$|z^{1/2}| = \sqrt{\rho},$$

and, if $\rho > 1$ we have that

$$|z^2+1| \ge ||z^2| - 1| = \rho^2 - 1.$$

Here we have used the fact that for two complex numbers $z_1$ and $z_2$ we have that by the triangle inequality,

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

Hence

$$\left|\frac{z^{1/2}}{z^2+1}\right| \leq \frac{\sqrt{\rho}}{\rho^2-1}.$$

In particular, by the Estimation Lemma, we have, for $\rho > 1$

$$\left|\int_{\gamma_\rho} \frac{z^{1/2}}{z^2+1}\right| \leq \frac{\sqrt{\rho}}{\rho^2-1} L(\gamma_\rho) = \pi\rho\frac{\sqrt{\rho}}{\rho^2-1} \to 0$$

as $\rho \to \infty$.

Q.7 *Let $\gamma$ be any piecewise $C^1$-curve from $-3$ to $3$ such that, except for the end points, lies entirely in the upper half plane. Calculate the integral*

$$\int_\gamma f(z)\,dz,$$

*where $f(z)$ is the branch of $z^{\frac{1}{2}}$ defined by $\sqrt{r}e^{i\theta/2}$ with $0 < \theta < 2\pi$.*

S.7 We note that the branch is not defined at the end points of the contour. This is a bit more general than the integrals we considered in class, since the curve is not entirely in the domain of definition of the function $f(z)$. However one can still make sense of it since, if we write $\gamma : [a, b] \to \mathbb{C}$ for a $C^1$ curve, then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

and the right hand side is still well-defined independently of the behaviour of $f$ at the end points (as long as $f$ is of course continuous on $\gamma$). Then we can extend the definition to contour (i.e. piecewise $C^1$-curves).

Going back to the question, we consider

$$f_1(z) = \sqrt{r}e^{i\theta/2} \quad \text{for} \quad z = re^{i\theta},$$

where $-\pi/2 < \theta < 3\pi/2$. Note that this is another branch of the square root, and is well defined on every contour lying above the real line (since the branch cut is in the lower half-plane). Moreover by the discussion above we have that

$$\int_\gamma f(z)\,dz = \int_\gamma f_1(z)\,dz$$

for any contour which lies above the real line, since $f$ and $f_1$ agree there. But now we can apply the FTC for $f_1$ since it is well defined at the end points. In particular , if we consider

$$F_1(z) := \frac{2}{3}z^{3/2} = \frac{2}{3}r\sqrt{r}e^{3i\theta/2}$$

with $-\pi/2 < \theta < 3\pi/2$ then $F_1'(z) = f_1(z)$. That is,

$$\int_\gamma f(z)\,dz = \int_\gamma f_1(z)\,dz = [F_1(z)]_{-3}^{3} = 2\sqrt{3}(1+i).$$