Complex Analysis II

Gappy Notes

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The following gappy notes are based on the official lecture notes, who have evolved over the years by Sabine Bögli, Thanasis Bouganis, Jens Funke, Katie Gittins, Stephen Harrap, and Michael Magee.
The LaTeX code of many images presented in this note is a modification of ones found online (such as in StackExchange) and with the help of ChatGPT.

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CHAPTER 1

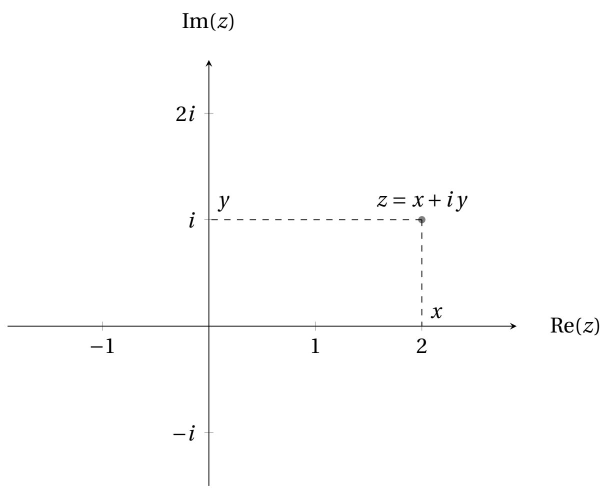
Complex numbers

Complex numbers are elements of the form

$$z = x + i y$$

where x and y are real numbers ($x, y \in \mathbb{R}$) and i is the *imaginary unit*.

We can visualise complex number as points in the plane $\mathbb{R}^2$ which we call an *Argand diagram*:



The set of all complex numbers is denoted by $\mathbb{C}$. Motivated by the above image, we sometime call $\mathbb{C}$ the *complex plane*. In the notation $z = x + iy$ we call x the *Real part of* z and y the *Imaginary part of* z which we denote as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively. In other words

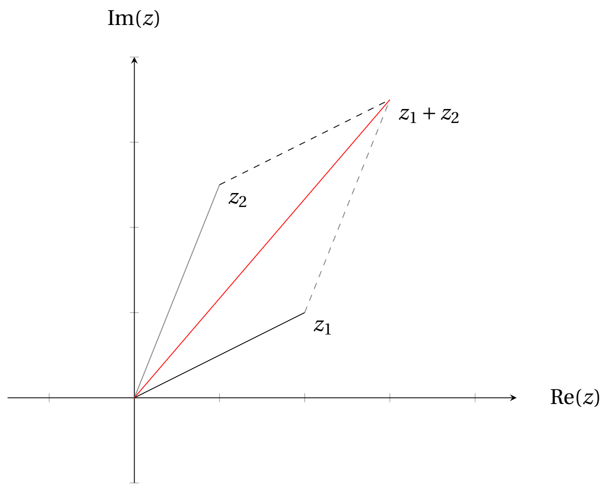
$$\mathrm{Re}(z) := x, \quad \mathrm{Im}(z) := y,$$

$$z = x + iy = \mathrm{Re}(z) + i\mathrm{Im}(z).$$

Addition. We can add/subtract two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the following way

$$z_1 \pm z_2 := (x_1 \pm x_2) + i(y_1 \pm y_2).$$

This has a simple geometric meaning in the complex plane - it is like adding two the vectors (x_1, y_1) and (x_2, y_2):



Multiplication. Multiplication of complex numbers follows "standard" multiplication on $\mathbb{R}$ with the caveat that $i^2 = -1$. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) =$$

What about division?

$$\left(x + iy\right)\left(x - iy\right) =$$

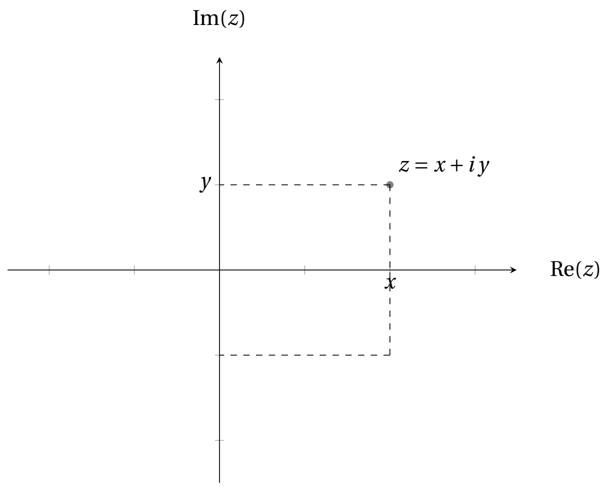
$$\overline{z} = x - iy = \mathrm{Re}\,(z) - i\mathrm{Im}\,(z)\,.$$

$$z\overline{z} = \mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2 \in \mathbb{R}$$

We denote by

$$|z| = \sqrt{z\overline{z}} = \sqrt{\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2}$$

and call it the *modulus of* z.



Using the conjugate we find that

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

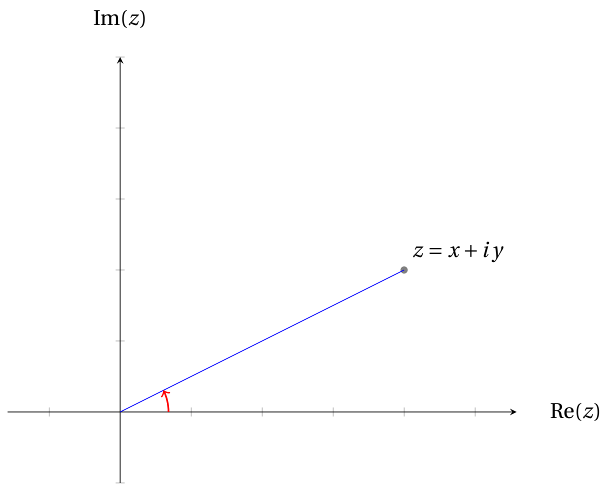
$$z^{-1} := \frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

EXAMPLE. Find $\frac{3}{1+i} - (3 + 2i)$.

LEMMA 1.1 (Important Properties of Complex numbers).

(1) $z_1z_2 = 0 \iff z_1 = 0$ *or* $z_2 = 0$.
(2) $|z| = \sqrt{z\overline{z}}$.
(3) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ *and* $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.
(4) $z^{-1} = \frac{\overline{z}}{|z|^2}$.

Polar coordinates.



where we define for any $z \neq 0$

$$|z| := r =$$

$$\arg(z) := \theta =$$

θ is called *the argument of z (denoted by* $\arg(z)$*)*.

Notation: The *principal value* of $\arg(z)$ is the value in the interval $(-\pi, \pi]$ and will be denoted $\text{Arg}(z)$.

Using polar coordinates we have

$$z = r\cos(\theta) + i r\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$$

EXAMPLE. Find $|z|$, $\mathrm{Arg}(z)$, and $\arg(z)$ for $z = 3i$ and $z = i + 1$.

LEMMA 1.2 (Properties of argument). *We have the following properties of the argument:*

(1) $\arg(z_1 z_2) = \big(\arg(z_1) + \arg(z_2)\big) \bmod 2\pi$
(2) $\arg(1/z) = -\arg(z) \bmod 2\pi$
(3) $\arg(\overline{z}) = -\arg(z) \bmod 2\pi$.

When we say two real numbers are equal $\bmod 2\pi$ *we mean they differ by an integer multiple of* 2π.

LEMMA 1.3. *If* $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ *and* $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$ *then*

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

PROOF.

□

Our above discussion also motivates the notation

$$e^{i\theta} := \cos(\theta) + i\sin(\theta).$$

and we find that

$$e^{i\theta_1} e^{i\theta_2} =$$

COROLLARY 1.4.

1. $|z_1 z_2| = |z_1||z_2|$.
2. De Moivre's formula:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

PROOF.

□

The modulus also has the following important properties.

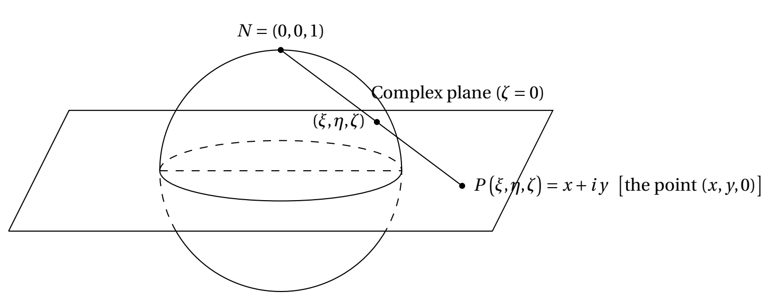
(1) (Triangle inequality) $|z_1 + z_2| \le |z_1| + |z_2|$
(2) $|z| \ge 0$ and $|z| = 0 \iff z = 0$.
(3) $\max(|\mathrm{Re}(z)|, |\mathrm{Im}(z)|) \le |z| \le |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

We can use the functions $|z|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $\arg(z)$ to describe various geometric domains in $\mathbb{C}$. For instance:

Functional expression	Domain in $\mathbb{C}$
$\mathbb{D} := \{z \in \mathbb{C} : \|z\| < 1\}$	
	Upper half plane (without the x axis)
$\mathbb{H}_R := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$	
	Left half plane (without the y axis)
$\arg(z) = \frac{\pi}{4}$	
	circle centred at i with radius 4

EXAMPLE. What is the set $|z - i| = |z + i|$?

The Riemann Sphere.



We can find a formula for P:

$$P(\xi,\eta,\zeta)=\frac{\xi}{1-\zeta}+i\frac{\eta}{1-\zeta}.$$

$$P^{-1}(z)=\left(\frac{2\mathrm{Re}\,(z)}{1+|z|^2},\frac{2\mathrm{Im}\,(z)}{1+|z|^2},\frac{|z|^2-1}{1+|z|^2}\right).$$

The stereographic map is a bijection between $\mathbb{C}$ and $\mathbb{S}^2\setminus\{N\}$.

A few examples to equivalent domains on $\mathbb{S}^2$ and $\hat{\mathbb{C}}$ via the stereographic map:

In $\mathbb{S}^2$	In $\hat{\mathbb{C}}$
$N=(0,0,1)$	
$S=(0,0,-1)$	
Equator	
	$\{z\in\mathbb{C}:\lvert z\rvert<1\}$
(open) Northern hemisphere	

DEFINITION 1.5. The ***Riemann sphere*** is the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ along with the stereographic projections from the north and south pole

Complex functions.

DEFINITION. Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$, be a function and let c be an interior point of X. Then f is called *continuous at* $c \in X$ if

$$\lim_{x \to c} f(x) = f(c).$$

That is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\big|f(x) - f(c)\big| < \varepsilon \quad \text{for all } x \in X \text{ with } |x - c| < \delta.$$

CHAPTER 2

Metric spaces

2.1. Metric spaces

DEFINITION 2.1 (Metric spaces). A metric space is a set X together with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that for all $x, y, z \in X$

- **(D1) Positivity.** $d(x, y) = 0 \iff x = y$.
- **(D2) Symmetry.** $d(x, y) = d(y, x)$.
- **(D3) Triangle inequality.** $d(x, y) \leq d(x, z) + d(z, y)$.

The function d is called a **metric** and we will often denote a metric space by (X, d). When the metric is clear from the context we sometimes only write X.

Examples of Metrics.

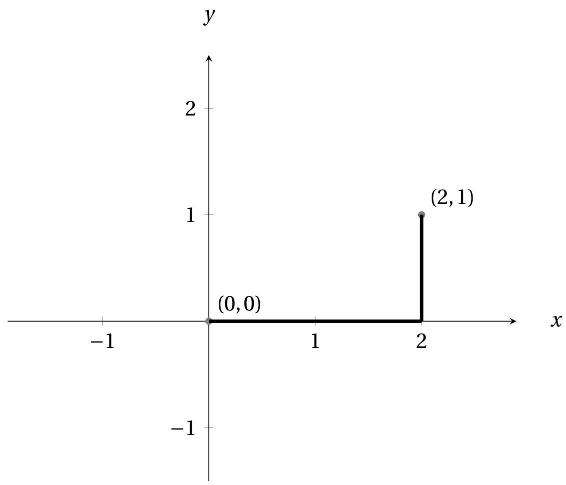
DEFINITION 2.2. [Norms and normed vector spaces] Given any real or complex vector space V, a function $\|.\|: V \to \mathbb{R}_{\geq 0}$ is a **norm** if it satisfies (for $v, w \in V$)

- **(N1)** $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.
- **(N2)** $\|\lambda v\| = |\lambda| \cdot \|v\| \quad$ for $\lambda \in \mathbb{R}$ or $\mathbb{C}$.
- **(N3)** $\|v + w\| \leq \|v\| + \|w\|$ (the triangle inequality).

A vector space equipped with a norm is called a **normed vector space**. Any norm induces a metric given by

$$d(v, w) := \|v - w\|.$$

EXAMPLE.



REMARK. Any non-zero subset $Y \subset X$ of a metric space (X, d) is itself a metric space with respect to the same metric (this is easy to check). Unless mentioned otherwise, this will always be the metric we will use on Y.

2.2. Open and closed sets

Recall that we say that a subset $X \subseteq \mathbb{R}$ is open if for any $c \in X$ there exists $\varepsilon > 0$ (that can depend on c!) such that

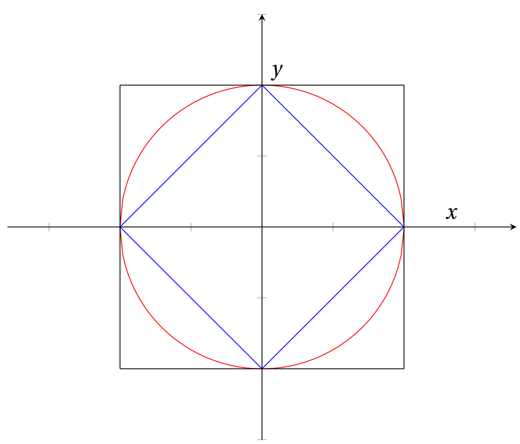
$$(c-\varepsilon, c+\varepsilon) \subseteq X.$$

DEFINITION 2.3 (Balls in a metric space). Let (X,d) be a metric space, $x \in X$ and let $r > 0$ be a real number. Then:

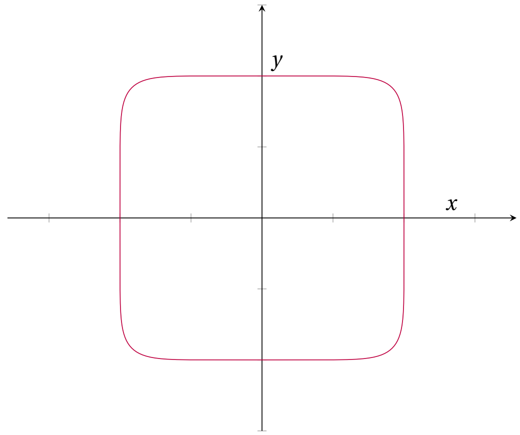
- The **open ball $B_r(x)$ of radius r centred at x** is
$$B_r(x) := \{y \in X : d(x,y) < r\}.$$
- The **closed ball $\overline{B}_r(x)$ of radius r centred at x** is
$$\overline{B}_r(x) := \{y \in X : d(x,y) \leq r\}.$$

EXAMPLE.

The following shows the open balls $B_1(0,0) \subset \mathbb{R}^2$ for the metrics induced by $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$:



As a bonus, here is what the open ball $B_1(0,0) \subset \mathbb{R}^2$ for the metric induced by $\|\cdot\|_4$ looks like:



DEFINITION 2.4. [Open/closed sets in a metric space] Let (X,d) be a metric space. Then:

- A subset $U \subseteq X$ is **open (in X)** if for every point $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.
- A subset $U \subseteq X$ is **closed (in X)** if its complement $U^c := X \setminus U$ is open.

LEMMA 2.5. *[Open balls are open] In a metric space, the open ball $B_r(x)$ is an open set.*

PROOF.

$\square$

REMARK. It can also be shown that in a metric space the closed ball $\overline{B}_r(x)$ is closed.

Open sets (examples/warnings).

Important observation: Open and closed sets are really *relative* notions, depending on the ambient space (as well as the metric).

Notation: When we say a subset of $\mathbb{R}$, or $\mathbb{R}^n$, or $\mathbb{C}$, is open/closed we mean with respect to the standard norms $|.|$, $\|.\|_2$ and $|.|$ respectively.

LEMMA 2.6. *[Unions and intersections of open sets] Let (X,d) be a metric space. Then:*

1. *Arbitrary unions of open sets are open; that is $\bigcup_{i\in I} U_i$ is open for any (possibly infinite) collection of open sets U_i.*
2. *Finite intersections of open sets are open; that is $\bigcap_{i=1}^n U_i$ is open for any finite collection of open sets U_i.*

PROOF.

□

COROLLARY 2.7 (Unions and intersections of closed sets). *Let* (X, d) *be a metric space. Then:*

1. *Finite unions of closed sets are closed.*
2. *Arbitrary intersections of closed sets are closed.*

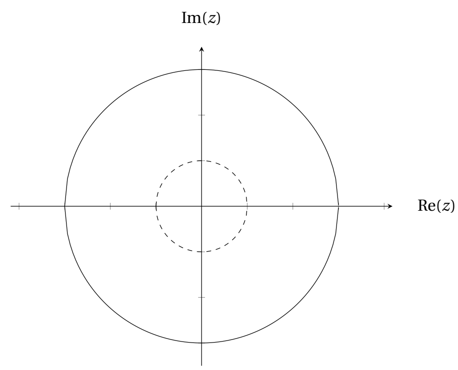
PROOF.

□

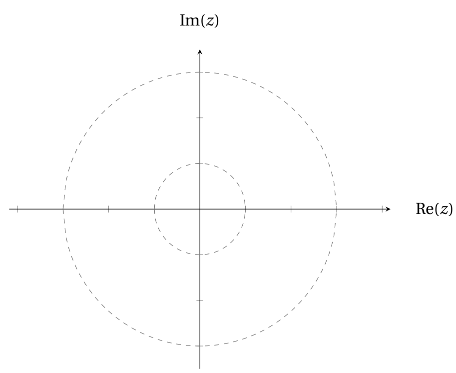
DEFINITION 2.8 (Interior points, closure, boundary). Let A be a subset of a metric space (X,d).

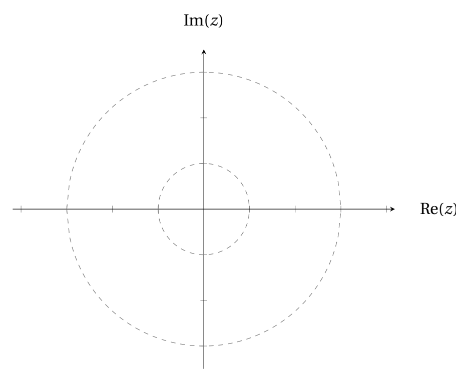
- The **interior** A^0 **of** A is defined by
$$A^0 := \{x \in A : \text{there exists an open set } U \subseteq A \text{ such that } x \in U\}.$$
- The **closure** $\overline{A}$ **of** A is the complement of the interior of the complement:
$$\overline{A} := \left(\left(A^c\right)^0\right)^c = \{x \in X : U \cap A \neq \varnothing \text{ for every open set } U \text{ with } x \in U\}.$$
- The **boundary** ∂A **of** A is the closure without the interior:
$$\partial A := \overline{A} \setminus A^0 \qquad \left[= \left(A^0\right)^c \cap \left(\left(A^c\right)^0\right)^c = \left(A^0 \cup \left(A^c\right)^0\right)^c\right].$$

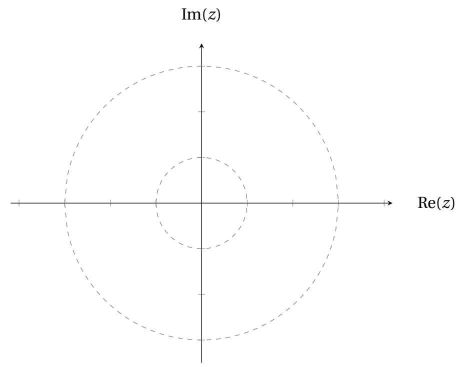
EXAMPLE. Let us consider the set
$$A = \{z \in \mathbb{C} : 1 < |z| \leq 3\}$$



Then







Important properties of a set A in a metric space (X,d):

1. A is open $\iff \partial A \cap A = \varnothing \iff A = A^0$.
   Moreover,
$$A^0 = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$$
2. A is closed $\iff \partial A \subseteq A \iff A = \overline{A}$.
   Moreover,
$$\overline{A} = \bigcap_{\substack{A \subseteq F \\ F \text{ closed}}} F.$$
3. $\partial A = \{x \in X : \text{for all open sets } U \text{ containing } x, \text{ there exist } y, z \in U \text{ with } y \in A \text{ and } z \in A^c\}$.

2.3. Convergence and continuity

DEFINITION 2.9. [Limits and convergence in a metric space] We say a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X,d) **converges to** $x \in X$ if we have

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

That is, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for every $n > N$. We write $x_n \to x$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = x$.

LEMMA. *A sequence of complex numbers $\{z_n\}$ converges in $(\mathbb{C}, |\cdot|)$ if and only if the sequences $\{\mathrm{Re}\,(z_n)\}$ and $\{\mathrm{Im}\,(z_n)\}$ converge in $(\mathbb{R}, |\cdot|)$.*

PROOF.

□

Important remark: Limits in the complex plane follow the COLT rules.

EXAMPLE. Consider the sequence $\{ik\}_{k\in\mathbb{N}}$ in $\hat{\mathbb{C}}$ and show that it converges to $\infty \in \hat{\mathbb{C}}$ with the chordal metric

$$d(z,w) = \left\| P^{1}(z) - P^{-1}(w) \right\|_2$$

where $P^{-1}(z) = \left(\frac{2\mathrm{Re}(z)}{1+|z|^2}, \frac{2\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2} \right)$.

LEMMA 2.10. *[Limits and open sets] Let (X,d) be a metric space. Then:*

1. A sequence can have at most one limit.
2. We have that $\lim_{n\to\infty} x_n = x$ if and only if for any open U with $x \in U$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have that $x_n \in U$.
   Hence the notion of a limit in a metric space can be stated in terms only of its open sets.

PROOF.

□

Sequences in metric spaces also give a nice criterion to check if a set is closed:

LEMMA 2.11. *[Closedness criterion in metric spaces] Let* (X, d) *be a metric space. Then* A *is a closed set if and only if for any sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* A *that converges to an element* $x \in X$ *we have that* $x \in A$.

PROOF.

□

Continuity.

DEFINITION 2.12 (Continuity). A map $f : (X_1, d_1) \to (X_2, d_2)$ between two metric spaces is called **continuous at** $x_0 \in X_1$ if for all $\varepsilon > 0$ there exists δ such that for all $x \in X_1$ with $d_1(x, x_0) < \delta$ we have that $d_2(f(x), f(x_0)) < \varepsilon$.
We say a function f is **continuous on** X_1 if it is continuous at every point $x_0 \in X_1$.

LEMMA 2.13 (Continuity via sequences). *A function* $f : X \to Y$ *between two metric spaces is continuous at* $x \in X$ *if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for every sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* X *such that* $\lim_{n\to\infty} x_n = x$.

PROOF. □

LEMMA 2.14. *[Basic properties of continuous functions]*

1. *Products, sum, quotients of real/complex valued continuous functions on a metric space* X *are continuous. E.g., if* $f : X \to \mathbb{C}$ *and* $g : X \to \mathbb{C}$ *are continuous, then* $f + g$ *and* fg *and* f/g *are continuous (where defined).*
2. *Compositions of continuous functions are continuous. I.e., if* $f : X_1 \to X_2$ *and* $g : X_2 \to X_3$ *are continuous maps between metric spaces, then* $g \circ f : X_1 \to X_3$ *is continuous.*

PROOF.

□

Reminder: For any function $f : X_1 \to X_2$ and any set $U \subseteq X_1$ the **preimage of U under f**, which we denote by $f^{-1}(U)$, is
$$f^{-1}(U) := \left\{x \in X_1 \,:\, f(x) \in U\right\}.$$

THEOREM 2.15. *[Continuity via open sets] Let X_1 and X_2 be metric spaces and let $f : X_1 \to X_2$ be a give map. Then the following are equivalent*

1. *f is continuous.*
2. *$f^{-1}(U)$ is open in X_1 for every open set $U \subseteq X_1$.*
3. *$f^{-1}(F)$ is closed in X_1 for every closed set $F \subseteq X_1$.*

PROOF.

□

EXAMPLE. Show that the set
$$U = \{(x, y) \in \mathbb{R}^2 \,:\, \left(x^2 + y^2\right) \sin^3\left(\sqrt{x^2 + 7}\right) > 2\}.$$
is open.

EXAMPLE. The set

$$U = \{(x, y) \in \mathbb{R}^2 \,:\, xy > 1,\, x^2 + y^2 > 3\}$$

is open.

Useful properties of preimage:

- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
- $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.
- Following from the above

$$f^{-1}(A^c) = f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A) = \left(f^{-1}(A)\right)^c.$$

2.4. Sequential Compactness and Compactness

DEFINITION 2.16 (Compactness). A non-empty subset K of a metric space X is called **sequentially compact** if for *any* sequence $\{x_n\}_{n\in\mathbb{N}}$ in K there exists a *convergent* subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ with limit in K.

PROPOSITION 2.17. *[Closed sets and limits of sequences] $F \subseteq X$ is closed if and only if every sequence in F which converges in X has its limit point in F. That is, if $\{x_n\}_{n\in\mathbb{N}}$ is in F and $\lim_{n\to\infty} x_n = x$ for some $x \in X$ then $x \in F$.*

PROOF.

□

COROLLARY 2.18. *[Relationship between sequential compactness and closedness]*

1. *Sequentially compact sets are closed.*
2. *Any closed subset of a sequentially compact subset is sequentially compact.*

LEMMA 2.19. *If $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence in a metric space X, then any subsequence of it converges to the same limit.*

PROOF.

□

PROOF OF THE RELATIONSHIP BETWEEN SEQUENTIAL COMPACTNESS AND CLOSEDNESS.

□

DEFINITION 2.20 (Bounded sets). A subset $A \subseteq X$ of a metric space X is said to be **bounded** if there exists $R > 0$ and $x \in X$ such that $A \subseteq B_R(x)$.

LEMMA 2.21. *[Sequentially compact sets are bounded] Let $K \subseteq X$ be a sequentially compact subset of a metric space X. Then K is bounded.*

PROOF.

□

THEOREM 2.22. *[Heine-Borel for $\mathbb{R}^n$ and $\mathbb{C}^n$] A subset K of $\mathbb{R}^n$ or $\mathbb{C}^n$ is sequentially compact (with respect to the standard metric) if and only if it is closed and bounded.*

PROOF.

□

THEOREM 2.23 (Extreme Value Theorem). *Let $f : X \to Y$ be a continuous map between two metric spaces. Then, if $K \subseteq X$ is sequentially compact the image $f(K)$ is sequentially compact in Y. In particular, for $Y = \mathbb{R}$, any continuous real-valued function on a metric space X attains minima and maxima on sequentially compact sets.*

PROOF.

□

COROLLARY 2.24. *If K is a closed an bounded set in $\mathbb{C}$ then $|z|$, $\mathrm{Re}(z)$, and $\mathrm{Im}(z)$ attain maximum and minimum on K.*

DEFINITION 2.25. Let X be a metric space. We say that a subset K is **compact** if whenever $\{U_i : i \in I\}$ is a collection of open subsets $U_i \subseteq X$ with $K \subseteq \bigcup_{i \in I} U_i$, then there exists a *finite* subset $J \subseteq I$ with $K \subseteq \bigcup_{i \in J} U_i$.

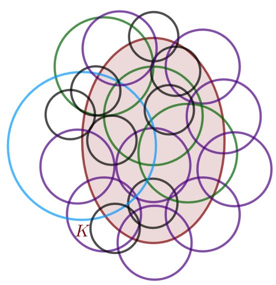


Figure. An “infinite” open cover of a set K on the left, and the finite open sub-cover of it on the right.

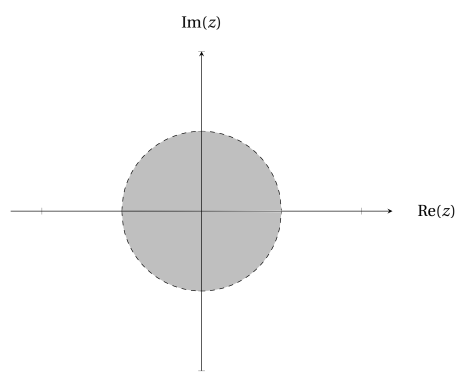
THEOREM 2.26. *Let X be a metric space and let K be a subset of X. Then K is sequentially compact if and only if K is compact.*

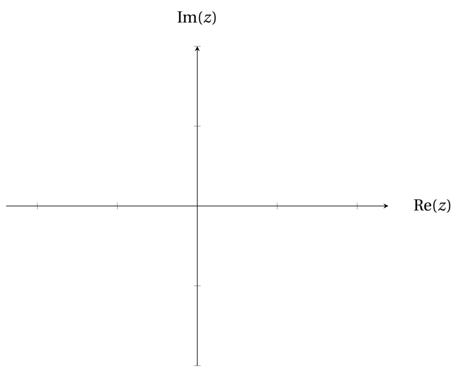
CHAPTER 3

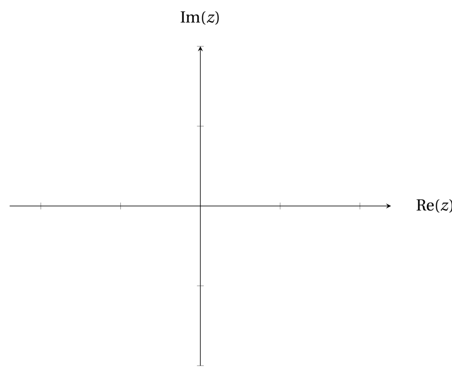
Complex functions and complex differentiation

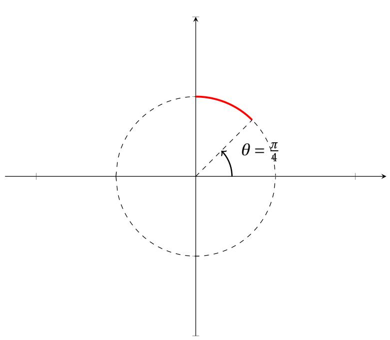
3.1. Visualising complex valued functions

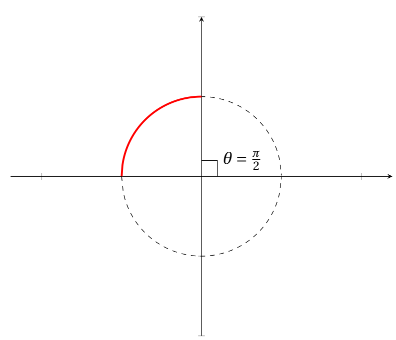
EXAMPLE. Consider the functions $f(z) = |z|$ and $g(z) = \arg(z)$. What are the images of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ under f and g?



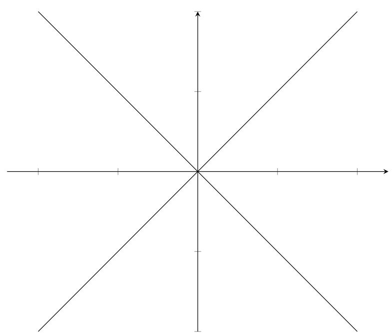

Im(z)
Re(z)

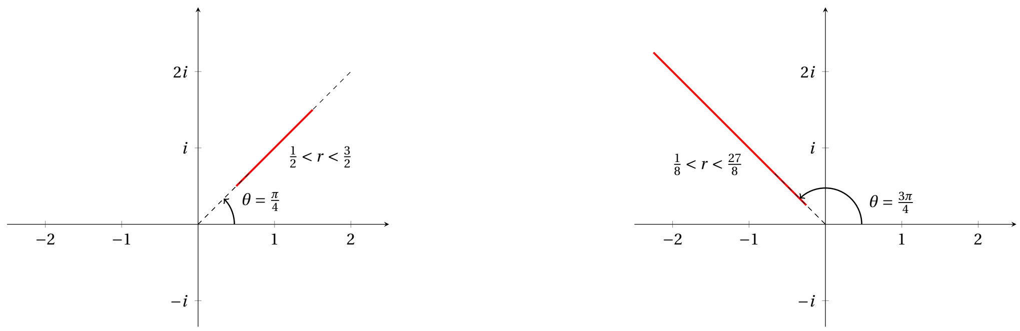

Im(z)
Re(z)


$\theta = \frac{\pi}{4}$


$\theta = \frac{\pi}{2}$

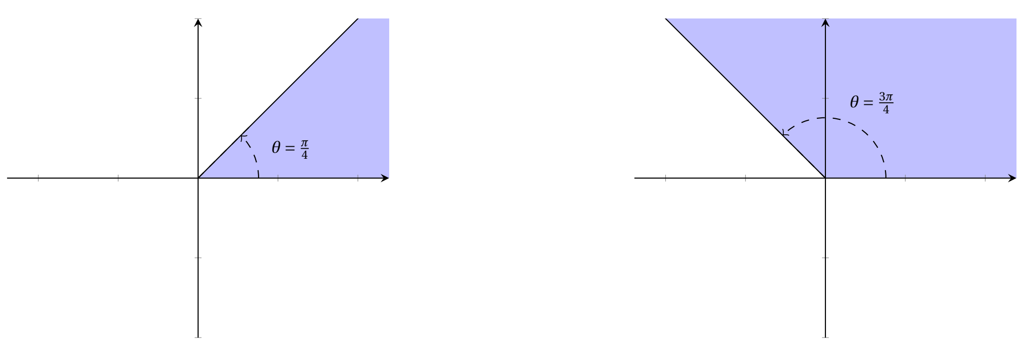
For instance, in the case that $n = 8$ we find





LEMMA.

- *The map* z^n *injectively takes an angular segment of length* $\frac{2\pi}{n}$ *which is open at one end and closed at the other from a circle of radius* r *to the entire circle of radius* r^n. *If the above segment is closed, or its size is larger than* $\frac{2\pi}{n}$ *the image is no longer injective.*
- *The map* z^n *injectively takes a ray of angle* θ *to a ray of angle* $n\theta \bmod 2\pi$. *Consequently, the map* z^n *injectively takes the wedge bounded by rays of angles* θ_1 *and* θ_2 *to the wedge bounded by rays of angles* $n\theta_1 \bmod 2\pi$ *and* $n\theta_2 \bmod 2\pi$ *if* $n|\theta_1-\theta_2| < 2\pi$. *When* $n|\theta_1-\theta_2| \geq 2\pi$ *the image is the entire complex plane (not invectively).*
- *We can define* n *different* $n-th$ *"roots" which are inverses to the map* z^n. *We can write them all in the form*
$$z^{\frac{1}{n}} = |z|^{\frac{1}{n}} e^{i\left(\frac{\operatorname{Arg}(z)}{n}+\frac{2\pi k}{n}\right)}$$
with $k = 0, \ldots, n-1$ *(we remove the* k *index from the map, but one need to specify which* k *we are talking about). Note that for a fixed* k *the* $n-th$ *root of* z *takes* $\mathbb{C}$ *into a anti clockwise rotation by* $\frac{\pi}{n}$ *of* $R_{n,k-1}$.



3.2. Exponential and trigonometric functions

DEFINITION 3.1 (Complex exponential). We define the **complex exponential function** $\exp : \mathbb{C} \to \mathbb{C}$ by

$$\exp(z) := e^x(\cos y + i \sin y). \qquad (z = x + iy)$$

As shorthand we write $\exp(z) = e^z$.

PROPOSITION 3.2. *We have the following properties of the complex exponential function:*

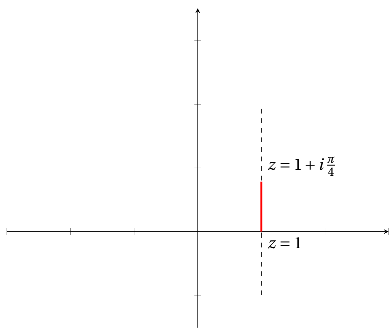
1. $e^z \neq 0 \quad$ *for all* $z \in \mathbb{C}$.
2. $e^{z_1+z_2} = e^{z_1} e^{z_2}$.
3. $e^z = 1$ *if and only if* $z = 2\pi i k$ *for some* $k \in \mathbb{Z}$.
4. $e^{-z} = 1/e^z$.
5. $|e^z| = e^{\mathrm{Re}(z)}$.

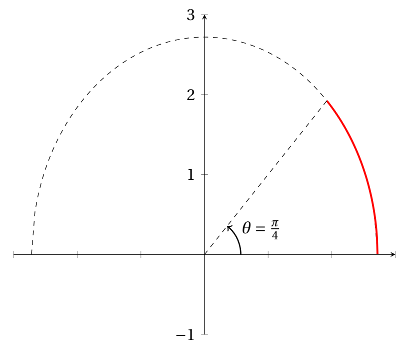
PROOF.

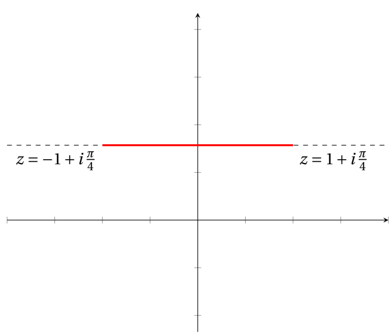
$\square$

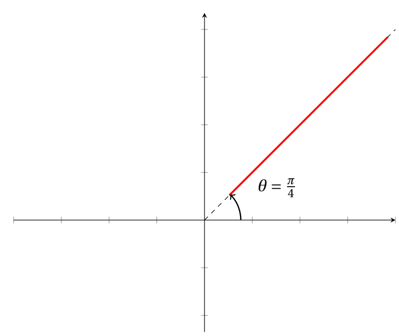
REMARK. A couple of observations:

- We have $\exp(2\pi i) = 1$ and $\exp(\pi i) = -1$. The latter is **Euler's formula**.
- The complex exponential function is $2\pi i$-periodic; that is, $\exp(z + 2k\pi i) = \exp(z)$ for any $k \in \mathbb{Z}$.









LEMMA.

- *The map e^z injectively takes a segment of length 2π which is open at one end and closed at the other from the line $x = c$ to the entire circle of radius e^c. If the above segment is closed, or its size is larger than 2π the image is no longer injective.*

- *The map e^z injectively takes the line $y = c$ to the ray of angle $c \mod 2\pi$ without the origin.*
- *The map e^z injectively takes the set $\{z \in \mathbb{C} : \theta < \operatorname{Im}(z) \le \theta + 2\pi\}$ to $\mathbb{C}^*$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.*

DEFINITION 3.3 (Trigonometric functions).

$$\sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \cos(z) := \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sinh(z) := \frac{1}{2}(e^{z} - e^{-z}) \qquad \cosh(z) := \frac{1}{2}(e^{z} + e^{-z})$$

LEMMA 3.4. *We have that for $z = x + iy$:*

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$
$$\sinh(z) = -i\sin(iz) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$
$$\cosh(z) = \cos(iz) = \cosh(x)\cos(y) + i\sinh(x)\sin(y).$$

In addition we have that for all $z \in \mathbb{C}$

$$\sin(z)^2 + \cos(z)^2 = 1, \qquad \cosh(z)^2 - \sinh(z)^2 = 1.$$

LEMMA.

- *The map $\sin(z)$ injectively takes a segment of length 2π which is open at one end and closed at the other from the line $y = c$ to an ellipse when $c \neq 0$. When $c = 0$ we get the "squished" ellipse $[-1, 1] \times \{0\}$ and the map is injective for a segment of length π on which the real map $\sin(x)$ is injective.*
- *The map $\sin(z)$ injectively takes the line $x = c$ to a one sided hyperbola when $c \neq \pi k$ and $c \neq \frac{\pi}{2} + \pi k$ for all $k \in \mathbb{Z}$. When $c = \pi k$ or $c = \frac{\pi}{2} + \pi k$ for some $k \in \mathbb{Z}$ we get the "squashed" hyperbolas:*
  - $\{0\} \times i\mathbb{R}$ *if $c = \pi k$. The map is injective in this case.*
  - $[1, \infty) \times \{0\}$ *if $c = \frac{\pi}{2} + 2\pi k$. The map is injective on $\left\{z \in \mathbb{C} : z = \frac{\pi}{2} + 2\pi k + iy,\ y \ge 0\right\}$ and $\left\{z \in \mathbb{C} : z = \frac{\pi}{2} + 2\pi k + iy,\ y \le 0\right\}$.*
  - $(-\infty, 1] \times \{0\}$ *if $c = -\frac{\pi}{2} + 2\pi k$. The map is injective on $\left\{z \in \mathbb{C} : z = -\frac{\pi}{2} + 2\pi k + iy,\ y \ge 0\right\}$ and $\left\{z \in \mathbb{C} : z = -\frac{\pi}{2} + 2\pi k + iy,\ y \le 0\right\}$.*

Similar statements can be done for $\cos(z)$ *using the identity* $\cos(z) = \sin\left(z + \frac{\pi}{2}\right)$ *(a shift by* $\frac{\pi}{2}$ *on the* $x-$*axis), and* $\sinh(z)$ *and* $\cosh(z)$ *using the identities*

$$\sinh(z) = -i\sin(iz), \qquad \cosh(z) = \cos(iz)$$

(rotations by $\pm\frac{\pi}{2}$ *of the variable and image in the complex plane).*

3.3. Logarithms and complex powers

LEMMA 3.5. *[Inverting the exponential function] For every* $w \in \mathbb{C}^*$, *the equation*

$$e^z = w \tag{3.1}$$

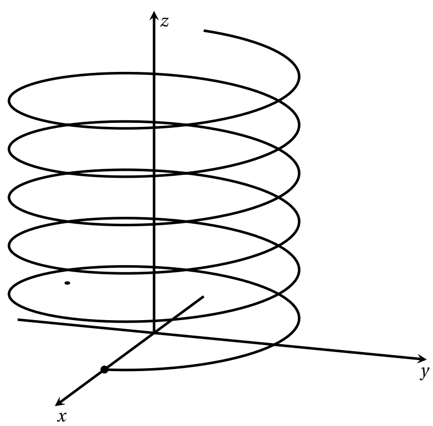
has a solution z. *Furthermore, if we write* $w = |w|e^{i\varphi}$ *with* $\varphi = \mathrm{Arg}(w)$, *then all solutions to* (3.1) *are given by*

$$z = \log|w| + i(\varphi + 2\pi k) \quad \text{for } k \in \mathbb{Z}. \tag{3.2}$$

Here, $\log|w|$ *is the usual natural logarithm of the real number* $|w|$. *Note that there are infinitely many solutions.*

PROOF.

$\square$



DEFINITION 3.6 (Complex logarithm functions). For any two real numbers $\theta_1 < \theta_2$ with $\theta_2 - \theta_1 = 2\pi$, let arg be the choice of argument function with values in $(\theta_1, \theta_2]$. Then the function

$$\log(z) := \log|z| + i\arg(z)$$

is called a ***branch of logarithm***. It has a jump discontinuity along the ray $R_{\theta_1} = R_{\theta_2}$. This ray is called a ***branch cut***.

If we choose $\arg(z) = \mathrm{Arg}(z) \in (-\pi, \pi]$, then we obtain a branch of logarithm called the ***principal branch of log***. We write Log for this principal branch: it is given by the formula

$$\mathrm{Log}(z) := \log|z| + i\mathrm{Arg}(z).$$

The principal branch of logarithm has a "jump discontinuity" along the ray given by the non-positive real axis $\mathbb{R}_{\leq 0}$.

LEMMA 3.7. *[Properties of logarithms] We have the following properties when using any given branch of logarithm:*

1. $e^{\log z} = z$ *for any* $z \in \mathbb{C} \setminus \{0\}$.

2. In general

$$\log(zw) \neq \log z + \log w.$$

3. In general

$$\log(e^z) \neq z.$$

PROOF.

□

LEMMA.

- *A branch of the map* $\log(z)$ *injectively takes a ray without the origin of angle* $\theta \neq 0$*, measured with respect to the branch cut, to the line* $y = \theta$*.*
- *A branch of the map* $\log(z)$ *takes a concentric circle of radius* r*, minus the branch cut, to a segment of length* 2π *on the line* $x = \log r$ *with its lowest point given by the angle that define the branch cut.*

DEFINITION 3.8 (Complex powers). For $w \in \mathbb{C}$ fixed, by choosing any branch of log we can define a branch of the function $z \mapsto z^w$ by the expression

$$z^w := \exp(w \log z).$$

For example, if $w = 1/n$ and we use the principal branch we get

$$z^{\frac{1}{n}} = e^{\frac{\log|z|}{n} + i\frac{\mathrm{Arg}(z))}{n}} = |z|^{\frac{1}{n}} e^{i\frac{\mathrm{Arg}(z)}{n}}.$$

EXAMPLE. Find $(1-i)^{\frac{1}{2}}$ using the principal branch of the logarithm.

EXAMPLE. Find $2^{\frac{1}{2}}$ for all possible branches.

3.4. Möbius transformations

3.5. Complex differentiability

DEFINITION 3.9 (Complex differentiability). A function $f : U \to \mathbb{C}$ defined on an *open* set U in $\mathbb{C}$ is **(complex) differentiable at** $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We call this limit the **derivative of f at z_0** and write $f'(z_0)$ for the limit, i.e.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Another form to the above limit is

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

EXAMPLE. Show that $f(z) = z^2$ is differentiable at $z = 0$.

EXAMPLE. At which points is $f(z) = \overline{z}$ differentiable?

LEMMA (COLT for derivatives and the chain rule).

1. *Let $f, g : \mathbb{C} \to \mathbb{C}$ be differentiable at $z_0 \in \mathbb{C}$. Then:*

- *$f+g$ is differentiable at z_0 and*
$$\left(f+g\right)'(z_0)=f'(z_0)+g'(z_0).$$
- *(product rule) fg is differentiable at z_0 and*
$$\left(fg\right)'(z_0)=f'(z_0)g(z_0)+f(z_0)g'(z_0).$$
- *(quotient rule) if $g(z_0)\neq 0$ then f/g is differentiable at z_0 and*
$$\left(\frac{f}{g}\right)'(z_0)=\frac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{g(z_0)^2}.$$

2. *(chain rule) Let $f,g:\mathbb{C}\to\mathbb{C}$ be functions such that g is differentiable at $z_0\in\mathbb{C}$ and f is differentiable at $g(z_0)$. Then $f\circ g$ is differentiable at z_0 and*
$$\left(f\circ g\right)'(z_0)=f'\left(g(z_0)\right)g'(z_0).$$

PROOF.

□

3.6. Cauchy-Riemann equations

PROPOSITION 3.10. *[Cauchy-Riemann equations] Let $f=u+iv$ be complex differentiable at $z_0=x_0+iy_0$. Then the real partial derivatives u_x,u_y,v_x,v_y exist at $\left(x_0,y_0\right)$ and satisfy the* ***Cauchy-Riemann equations****:*
$$u_x\left(x_0,y_0\right)=v_y\left(x_0,y_0\right)\qquad\qquad u_y\left(x_0,y_0\right)=-v_x\left(x_0,y_0\right).$$
Furthermore, the derivative of f at z_0 can be written as
$$\begin{aligned}f'(z_0)&=u_x(z_0)+iv_x(z_0)=v_y(z_0)-iu_y(z_0)\\&=u_x(z_0)-iu_y(z_0)=v_y(z_0)+iv_x(z_0).\end{aligned}$$

PROOF.

□

THEOREM 3.11. *Let $f = u + iv$ be defined on an open subset U of $\mathbb{C}$. Assume the partial derivatives u_x, u_y, v_x, v_y exist, are continuous, and satisfy the Cauchy-Riemann equations at $z_0 \in U$. Then f is complex differentiable at z_0.*

PROOF.

□

Holomorphicity.

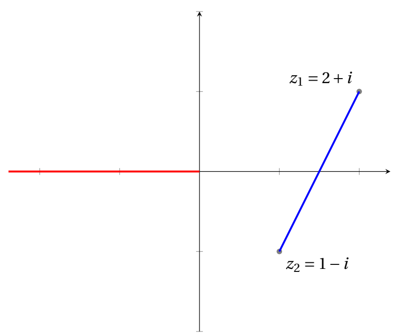
DEFINITION 3.12 (Holomorphic functions). A function $f : U \to \mathbb{C}$ defined on an open set $U \subseteq \mathbb{C}$ is **holomorphic on** U if it is complex differentiable at every point in U. We say f is **holomorphic at** z_0 if it is holomorphic on a open ball $B_\varepsilon(z_0)$ for some $\varepsilon > 0$.

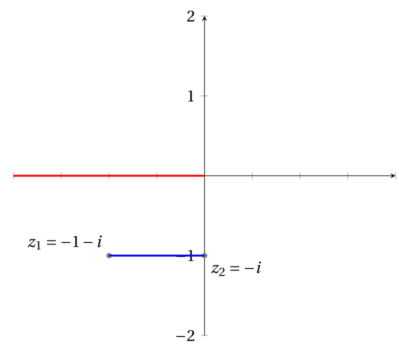
DEFINITION 3.13. [Paths & path-connectedness]

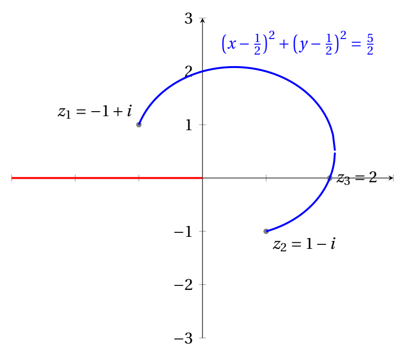
1. A **path or curve from** $z \in \mathbb{C}$ **to** $w \in \mathbb{C}$ is a continuous function $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = w$. We say the path/curve is **closed** if $z = w$ (in this case, the endpoints of the path join up).
2. A path/curve is said to be **continuously differentiable, or** C^1**,** if its real part and imaginary parts are continuously differentiable on $[0,1]$. At the end point 0 and 1 this means that real part and imaginary parts have right-sided derivatives at 0 and left-sided derivative at 1, and that the derivatives are continuos from the right at 0 and from the left at 1.
   In that case we define
   $$\gamma'(t) = \left(\mathrm{Re}\left(\gamma(t)\right)\right)' + i\left(\mathrm{Im}\left(\gamma(t)\right)\right)'.$$
3. We say a subset $U \subseteq \mathbb{C}$ is C^1 **path-connected** if for every pair of points $z, w \in U$ there exists a C^1 path from z to w such that $\gamma(t) \in U$ for all $t \in [0,1]$. For simplicity, we will use the term path-connected instead of C^1 path connected in the remaining of this module.

In general paths can be defined from an interval $[a,b]$ instead of $[0,1]$. It makes no difference since we can always re-parametrise the paths. For instance, if $\gamma_1 : [a,b] \to X$ is a given function we just define $\gamma : [0,1] \to X$ by
$$\gamma(t) = \gamma_1\left(\frac{t-a}{b-a}\right).$$
Conversely for $\gamma : [0,1] \to X$ we define $\gamma_1 : [a,b] \to X$ by
$$\gamma_1(t) = \gamma\left((b-a)\,t + a\right).$$
Note that the continuity or differentiability of the maps is identical, as well as their image, so our definition of paths and C^1 paths remains intact.

DEFINITION 3.14 (Domains). A **domain** D is an open, path-connected subset of $\mathbb{C}$. (Some people call domains **regions**).







THEOREM 3.15. *Let $f : D \to \mathbb{C}$ be holomorphic on a domain $D \subseteq \mathbb{C}$. If $f'(z) = 0$ for every $z \in D$ then f is constant on $\mathbb{D}$.*

LEMMA 3.16. *[Chain rule] Let $U \subseteq \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a holomorphic function on $\mathbb{C}$ and $\gamma : [0,1] \to U$ be a C^1 path. Then for $t_0 \in [0,1]$ we have*

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0).$$

PROOF OF THE LEMMA.

□

PROOF OF THE THEOREM.

□

3.7. The angle-preserving properties of holomorphic functions

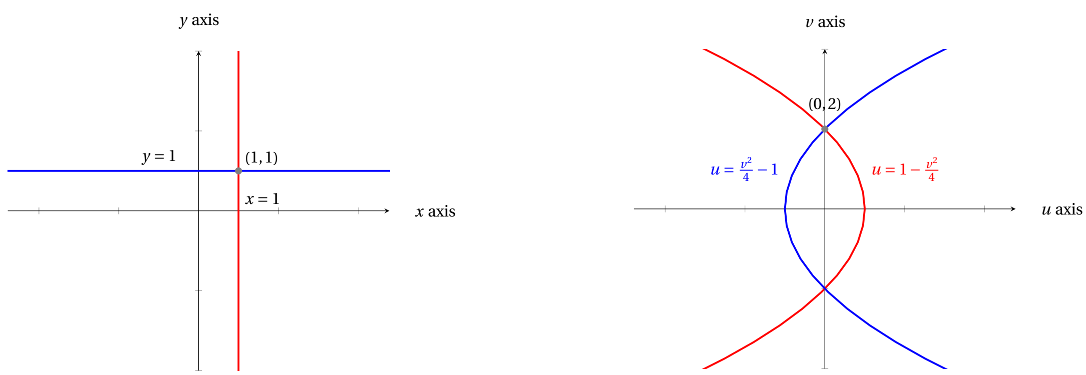
DEFINITION 3.17 (Conformal maps). We say that a (real differentiable) map $f : D \to \mathbb{C}$ on a domain $D \subseteq \mathbb{C}$ is **conformal at** z_0 if it preserves the angle and orientation between any two tangent vectors at z_0. This is exactly the same as saying that it preserves the angle and orientation between any two C^1 curves passing through z_0. We say that f is **conformal** if it is conformal at all points in D.

LEMMA 3.18. *[Holomorphic maps are conformal] Let f be a holomorphic map at z_0. If $f'(z_0) \neq 0$ then f conformal at z_0.*

PROOF.

$\square$

COROLLARY 3.19. *Any conformal map maps orthogonal grids in the (x, y)-plane to orthogonal grids.*



PROPOSITION 3.20. *[Conformal maps are holomorphic] Let D be a domain. If f is conformal at $z_0 \in D$ then f is complex differentiable at z_0 and $f'(z_0) \neq 0$. Therefore, if f is conformal on D, then f is holomorphic on D and $f'(z) \neq 0$ for all $z \in D$. Thus*

$$f \text{ is conformal on } D \quad \Longleftrightarrow \quad f \text{ is holomorphic with } f'(z) \neq 0 \text{ for all } z \in D.$$

PROOF.

$\square$

3.8. Biholomorphic maps

DEFINITION 3.21 (Biholomorphic maps). Let D and D' be domains. We say that $f : D \to D'$ is **biholomorphic** if f is holomorphic, a bijection, and the inverse $f^{-1} : D' \to D$ is also holomorphic. A biholomorphic map f is called a **biholomorphism**. When f as above exists, we say that the domains D and D' are **biholomorphic** and write $f : D \xrightarrow{\sim} D'$.

LEMMA 3.22 (Automorphism groups). *Let $D \subset \mathbb{C}$ be a domain. The set of all biholomorphic maps $f : D \xrightarrow{\sim} D$ from D to itself forms a group under composition. We call this group the* ***automorphism group of*** *D and denote it by* $\mathrm{Aut}(D)$.

PROOF.

$\square$

CHAPTER 4

Möbius transformations

4.1. Definition and first properties of Möbius transformations

Recall the that the *General Linear group* $\mathrm{GL}_2(\mathbb{C})$ is defined to be

$$\mathrm{GL}_2(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\},$$

i.e. the set of all 2×2 complex valued matrices with non-zero determinant.

DEFINITION 4.1 (Möbius transformations). Given any matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ we can define a function

$$M_T : \mathbb{C} \to \hat{\mathbb{C}}$$

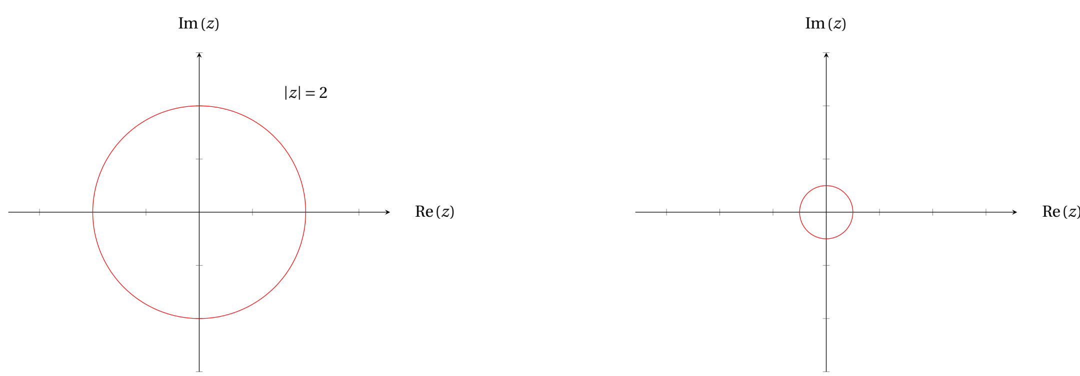
by

$$M_T(z) := \frac{az + b}{cz + d}$$

if $cz+d \neq 0$, and if $cz+d=0$ we set $M_T\left(-\frac{d}{c}\right)=\infty$ when $c \neq 0$. We can extend the map to $\hat{\mathbb{C}}$ by setting

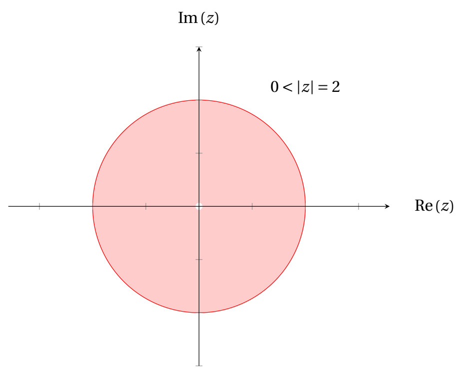
$$M_T(\infty)=\begin{cases}\frac{a}{c}, & \text{if } c \neq 0,\\ \infty, & \text{if } c=0.\end{cases}$$

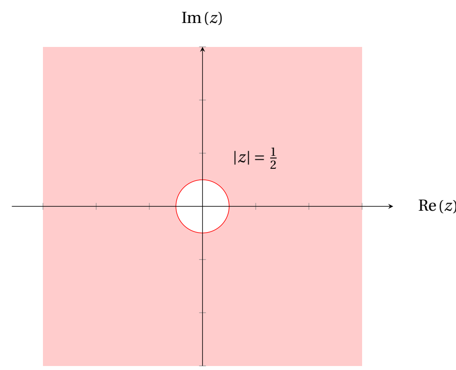
The function $M_T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called a **Möbius transformation**.



LEMMA.

- *The map $1/z$ injectively takes the set $\mathbb{D}\setminus\{0\} = \{z\in\mathbb{C} : 0<|z|<1\}$ to $\mathbb{D}^c = \{z\in\mathbb{C}\,|:|z|>1\}$. When considering $1/z$ over $\hat{\mathbb{C}}$ we find that it injectively takes $\mathbb{D}$ to $\mathbb{D}^c\cup\{\infty\}$. Similarly, $1/z$ injectively takes the set $\mathbb{D}^c$ to $\mathbb{D}\setminus\{0\}$ and on $\hat{\mathbb{C}}$ it takes $\mathbb{D}^c\cup\{\infty\}$ to $\mathbb{D}$.*
- *The map $1/z$ injectively takes the ray at angle θ without the origin to the ray at angle $-\theta$ without the origin. On $\hat{\mathbb{C}}$ the statement remains the same by adding the origin and ∞ to the rays.*





LEMMA 4.2. *The set of Möbius transformations form a group under composition. Furthermore,*

1. $M_{T_1}\circ M_{T_2} = M_{T_1T_2}$.
2. $(M_T)^{-1} = M_{T^{-1}}$.
3. *For any $k\in\mathbb{C}^*$ we have that $M_{kT} = M_T$. We conclude that $M_T = \mathrm{Id}$ if and only if*

$$T = k\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

for some $k\in\mathbb{C}^*$.

PROOF. $\square$

COROLLARY 4.3. *Any Möbius transformation is a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.*

LEMMA 4.4. *Let $T = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbb{C})$. If $c=0$, the Möbius transformation M_T gives a biholomorphic map*

$$M_T : \mathbb{C}\xrightarrow{\sim}\mathbb{C}.$$

If $c\neq 0$, then M_T gives a biholomorphic map

$$M_T : \mathbb{C}\setminus\left\{\frac{-d}{c}\right\}\xrightarrow{\sim}\mathbb{C}\setminus\left\{\frac{a}{c}\right\}.$$

PROOF.

$\square$

COROLLARY 4.5. *A Möbius transformation M_T is conformal at all $z \in \mathbb{C}$ with $M_T(z) \neq \infty$.*

4.2. How to find Möbius transformations

DEFINITION 4.6. Given four distinct points $z_0, z_1, z_2, z_3 \in \mathbb{C}$, the *cross-ratio* of these points is defined by

$$(z_0, z_1; z_2, z_3) := \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)} = \frac{\frac{z_0 - z_2}{z_0 - z_3}}{\frac{z_1 - z_2}{z_1 - z_3}}.$$

We will denote the cross ratio of z_0, z_1, z_2, z_3 by $(z_0, z_1; z_2, z_3)$. We can extend the definition to the case that one of the points is ∞ by removing all differences involving that point, for example,

$$(\infty, z_1; z_2, z_3) := \frac{(z_1 - z_3)}{(z_1 - z_2)}.$$

THEOREM 4.7. *[Three points Theorem] Let $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ be two sets of three ordered distinct points in $\hat{\mathbb{C}}$. Then there exists a unique Möbius Transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.*

LEMMA 4.8 (Fixed points). *Let $T \in \mathrm{GL}_2(\mathbb{C})$. If $M_T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is not the identity map then M_T has at most 2 fixed points in $\hat{\mathbb{C}}$, where z_0 is a fixed point of the map f if $f(z_0) = z_0$. In other words, if a Möbius transformation has three fixed points in $\hat{\mathbb{C}}$, then it is the identity.*

PROOF OF FIXED POINT LEMMA.

□

PROOF OF THREE POIINTS THEOREM.

□

LEMMA 4.9. *The cross-ratio is preserved under Möbius transformations. In other words, if $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ is a Möbius transformation then*

$$(f(z_0), f(z_1); f(z_2), f(z_3)) = (z_0, z_1; z_2, z_3).$$

LEMMA 4.10 (Building blocks of Möbius transformations). *Any Möbius transformation is composition of the following four type of maps:*

1. *Shift: $z\to z+b$ for some $b\in\mathbb{C}$.*
2. *Stretch/Compression: $z\to\lambda z$ for some $\lambda\in\mathbb{R}_{>0}$.*
3. *Rotation: $z\to e^{i\theta}z$ for some $\theta\in(-\pi,\pi]$.*
4. *Inversion: $z\to\frac{1}{z}$.*

PROOF.

□

PROOF OF THE INVARAINCE OF THE CROSS-RATIO.

$\square$

Using the invariance of cross-ratio to find Möbius transformation.

EXAMPLE. Find the Möbius map that takes the points $\{1,-1,i\}$ to $\{0,\infty,1\}$ respectively.

4.3. The image of special domains under the Möbius transformation - the geometry of circles and lines

LEMMA 4.11. *Let* $T=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ *and let* $D\subseteq\mathbb{C}$ *be a open set. Then* $M_T\left(D\setminus\left\{-\frac{d}{c}\right\}\right)$ *is open. Together with the fact that Möbius maps are bijections on* $\hat{\mathbb{C}}$ *we have that*

$$M_T\left(\partial D\setminus\left\{-\frac{d}{c}\right\}\right)=\partial M_T(D)\setminus\{\infty\}.$$

When $c=0$ *we consider* $-\frac{d}{c}$ *as* ∞ *which is outside of* $\mathbb{C}$ *and consequently* $A\setminus\{\infty\}=A$ *for any* $A\subseteq\mathbb{C}$.

LEMMA 4.12. *The map $f(z) = \frac{1}{z}$ takes a circle that doesn't pass through the origin to a circle. Moreover, if B is an open ball whose closure doesn't contain the origin then $f(B)$ is once again an open ball. Consequently, if $D \subseteq \mathbb{C}$ is a set then for any $z_0 \in \mathbb{D} \setminus \{0\}$ for which there exists $\varepsilon > 0$ with $B_\varepsilon(z_0) \subseteq D$ we have that there exists $\delta > 0$ such that $B_\delta\left(f(z_0)\right) \subseteq f(D)$ (i.e. f takes any non-zero interior point of D to an interior point of $f(D)$).*

PROOF OF THE AUXILARY LEMMA.

□

PROOF OF THE LEMMA.

□

LEMMA 4.13. *[Equation of circles and lines in $\mathbb{C}$] Given $\gamma, \beta \in \mathbb{R}$ and $\alpha \in \mathbb{C}$, the equation*

$$\gamma z\overline{z} - \alpha\overline{z} - \overline{\alpha}z + \beta = 0$$

describes a circle if $\gamma \neq 0$ and $|\alpha|^2 - \beta\gamma > 0$, and a line if $\gamma = 0$ and $\alpha \neq 0$. Conversely, any circle or line can be described by an equation of this form.

PROPOSITION 4.14. *Möbius transformations map circles and lines in $\hat{\mathbb{C}}$ to circles and lines in $\hat{\mathbb{C}}$, where we consider any line to pass through infinity. By circles in $\hat{\mathbb{C}}$ we mean simply circles in $\mathbb{C}$.*

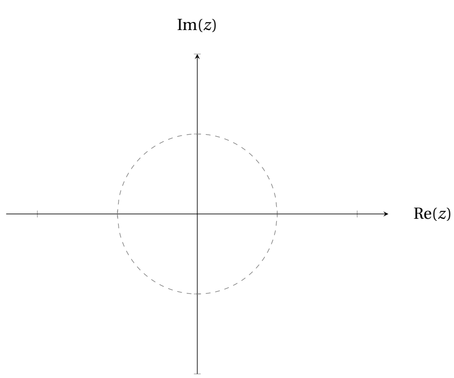
PROOF.

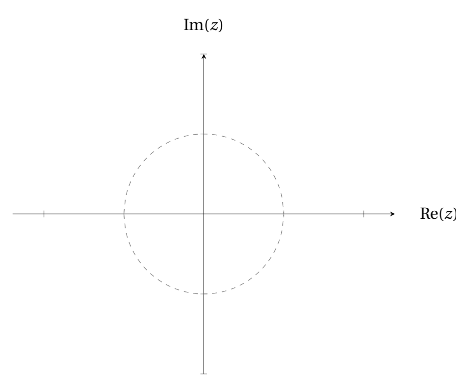
$\square$

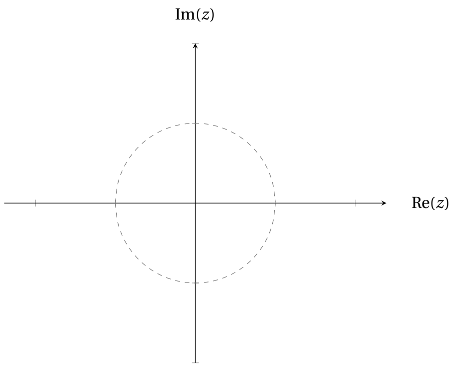
Terminology: we use the term **circline** to refer to an object that is either a circle or line.

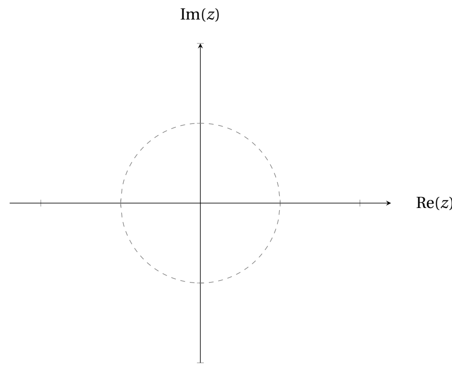
EXAMPLE. Find the image of $\mathbb{D}$ and $\mathbb{H}$ under the **Cayley map** defined as

$$f(z) = \frac{z-i}{z+i}.$$

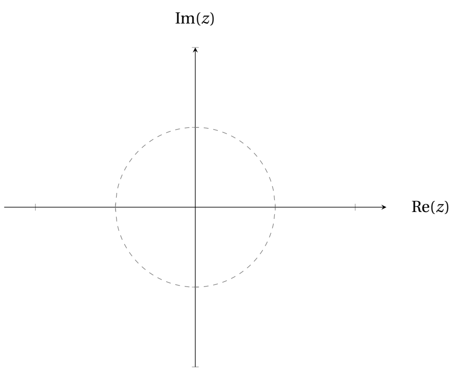


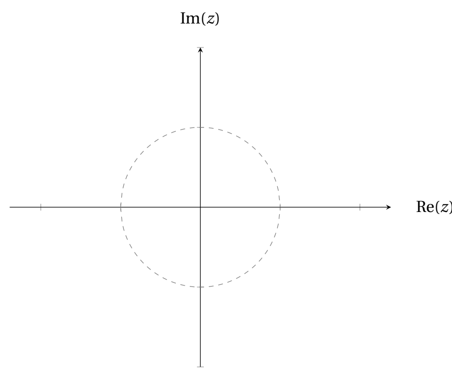






EXAMPLE. Show that the Cayley map takes $\left\{z \in \mathbb{C} : 0 < \operatorname{Arg}(z) < \frac{\pi}{2}\right\}$ to $\{z \in \mathbb{D} : \operatorname{Im}(z) < 0\}$.





The above examples also give us maps from discs/half discs to upper/lower/left/right planes and quadrants by using the inverse Cayley map

$$(M_C)^{-1}(z) = M_{C^{-1}}(z) = \frac{iz+i}{1-z}.$$

Möbius transformations that preserve the upper plane and unit disc.

PROPOSITION 4.15. *[H2H] Every Möbius transformation mapping $\mathbb{H}$ to $\mathbb{H}$ is of the form M_T with T in the group*

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, a, b, c, d \in \mathbb{R},\ \det T = ad - bc = 1 \right\}.$$

Conversely, every such Möbius transformation maps $\mathbb{H}$ to $\mathbb{H}$, and hence gives a biholomorphism from $\mathbb{H}$ to $\mathbb{H}$.

PROOF.

□

PROPOSITION 4.16. *[D2D] Every Möbius transformation from the unit disk $\mathbb{D}$ to itself is of the form M_T with T in the set*

$$\mathrm{SU}(1,1) := \left\{ T = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \det T = |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Conversely, every such Möbius transformation maps $\mathbb{D}$ to $\mathbb{D}$ and hence gives a biholomorphic automorphism of $\mathbb{D}$.

PROOF.

□

COROLLARY 4.17 (D2D-improved).

1. *Every Möbius transformation f from the unit disk $\mathbb{D}$ to itself can be written as*
$$f(z) = e^{i\theta}\,\frac{z - z_0}{\overline{z_0}z - 1},$$
for some angle θ and $z_0 \in \mathbb{D}$ which is the unique point such that $f(z_0) = 0$.
2. *All Möbius transformations of the unit disk to itself for which $f(0) = 0$ are rotations about 0.*

PROOF.

□

EXAMPLE. Find a biholomorphic map f from the unit disc to itself such that $f\left(\frac{i}{2}\right) = 0$ and $f(-i) = 1$.

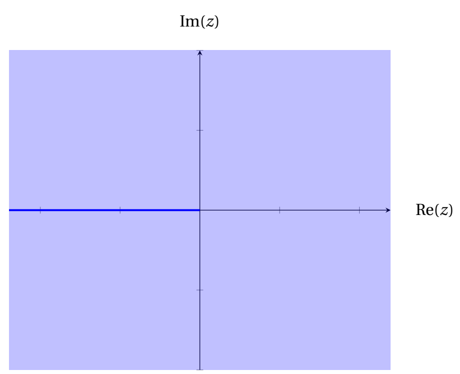
4.4. Riemann Sphere - revisited

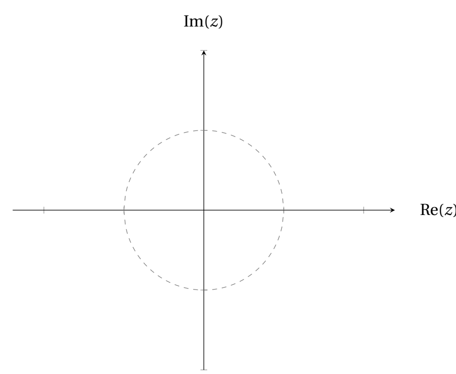
4.5. Biholomorphic domains - revisited

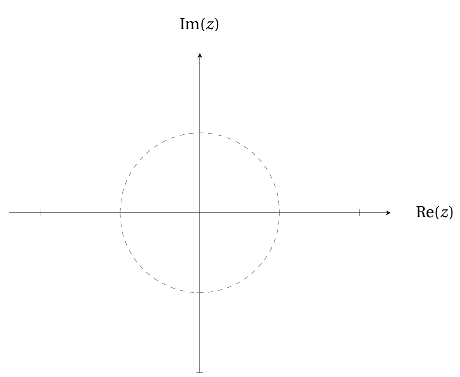
Function	Maps the domain	To the domain	Biholomorphic?
$z^n, \quad n \in \mathbb{N}$	$\{0 < \operatorname{Arg}(z) < \theta\} \bigcup \{0\}$	$\mathbb{C}$, if $\theta > \frac{2\pi}{n}$, $\{0 < \operatorname{Arg}(z) < n\theta\} \bigcup \{0\}$, if $\theta \le \frac{\pi}{n}$, $\{0 < \operatorname{Arg}(z) \le \pi\} \bigcup \{-\pi < \operatorname{Arg}(z) < n\theta - 2\pi\} \bigcup \{0\}$, if $\frac{\pi}{n} < \theta \le \frac{2\pi}{n}$.	When $\theta \le \frac{2\pi}{n}$.
$z^n, \quad n \in \mathbb{N}$	$\{\theta_1 < \operatorname{Arg}(z) < \theta_2\}$ with $-\pi < \theta_1 < \theta_2 \le \pi$	a rotation by $n\theta_1$ of the image of $\{0 < \operatorname{Arg}(z) < \theta_2 - \theta_1\}$ by z^n	When $\theta_2 - \theta_1 \le \frac{2\pi}{n}$.
Principle branch of $z^{\frac{1}{n}}$, $n \in \mathbb{N}$	$\{\theta_1 < \operatorname{Arg}(z) < \theta_2\} \bigcup \{0\}$ with $-\pi < \theta_1 < \theta_2 \le \pi$	$\left\{\frac{\theta_1}{n} < \operatorname{Arg}(z) < \frac{\theta_2}{n}\right\} \bigcup \{0\}$	Yes
e^z	$\{\theta_1 < \operatorname{Im}(z) < \theta_2\}$	$\mathbb{C}^*$, if $\theta_2 - \theta_1 > 2\pi$, $\{\theta_1 \bmod 2\pi < \operatorname{Arg}(z) < \theta_2 \bmod 2\pi\} \setminus \{0\}$, if $\theta_2 - \theta_1 \le 2\pi$.	When $\theta_2 - \theta_1 \le 2\pi$
$\operatorname{Log}(z)$	$\{\theta_1 < \operatorname{Arg}(z) < \theta_2\} \setminus \{0\}$ with $-\pi < \theta_1 < \theta_2 \le \pi$	$\{\theta_1 < \operatorname{Im}(z) < \theta_2\} \setminus \{0\}$	Yes
$\frac{1}{z}$	$\{r_1 < \lvert z\rvert < r_2\}$	$\left\{\frac{1}{r_2} < \lvert z\rvert < \frac{1}{r_1}\right\}$	Yes
Cayley map $f(z) = \frac{z-i}{z+i}$	$\mathbb{D}$	$\mathbb{H}_L$	Yes
Cayley map $f(z) = \frac{z-i}{z+i}$	$\mathbb{H}$	$\mathbb{D}$	Yes
Cayley map $f(z) = \frac{z-i}{z+i}$	$\mathbb{H} \cap \mathbb{D} = \{\lvert z\rvert < 1, \operatorname{Im}(z) > 0\}$	$\mathbb{H}_L \cap \mathbb{D} = \{\lvert z\rvert < 1, \operatorname{Re}(z) < 0\}$	Yes
Cayley map $f(z) = \frac{z-i}{z+i}$	$\mathbb{H} \cap \mathbb{H}_R = \{\operatorname{Re} z > 0, \operatorname{Im}(z) > 0\}$	$\mathbb{H}_- \cap \mathbb{D} = \{\lvert z\rvert < 1, \operatorname{Im}(z) < 0\}$	Yes
Möbius map $M_T(z)$ with real matrix T such that $\det T = 1$	$\mathbb{H}$	$\mathbb{H}$	Yes
$f(z) = e^{i\theta} \frac{z - z_0}{\overline{z_0} z - 1}$ with $\theta \in (-\pi, \pi]$ and $z_0 \in \mathbb{D}$	$\mathbb{D}$	$\mathbb{D}$	Yes
General Möbius transformation	domains bounded by circlines	domains bounded by circline	Yes

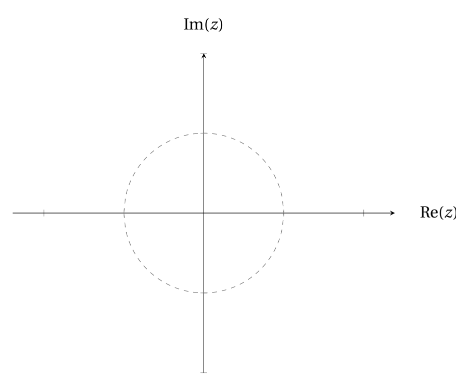
where we have used the notation of $\mathbb{H}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ for the lower half plane.

EXAMPLE. Find a biholomorphic map that takes $\{z \in \mathbb{C} : \operatorname{Arg}(z) \neq -\pi\}$ to $\mathbb{D}$.









CHAPTER 5

Notions of Convergence in complex analysis and power series

DEFINITION 5.1 (Pointwise convergence). Let (X, d_X) and (Y, d_Y) be two metric spaces. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ **converges pointwise (on X) to** f if every $x \in X$ the **limit function** $f(x) := \lim_{n\to\infty} f_n(x)$ exists in Y. In other words, for any $x \in X$ and any $\varepsilon > 0$ there exists $N(\varepsilon, x) \in \mathbb{N}$ such that if $n > N(\varepsilon, x)$

$$d_Y\left(f_n(x), f(x)\right) < \varepsilon.$$

Note that $N(\varepsilon, x)$ depends on ε and $x \in X$ in general.

Key issue with pointwise convergence: Pointwise convergence doesn't necessarily preserves continuity.

DEFINITION 5.2 (Uniform convergence). We say a sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to Y$ **converges uniformly (on X) to (the limit function)** f if we have

$$\sup_{x\in X} d_Y\left(f_n(x), f(x)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

In other words, for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $n > N(\varepsilon)$

$$d(f_n(x), f(x)) < \varepsilon, \qquad \forall x \in X.$$

Note that $N(\varepsilon)$ here *does not* depend on the specific choice of $x \in X$ – the same N works for all of them!

THEOREM 5.3. *[Uniform limits of continuous functions are continuous] Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\{f_n\}_{n\in N} : X \to Y$ be a sequence of continuous functions that converges uniformly to f on X. Then f is continuous on X.*

PROOF.

□

LEMMA 5.4. *[Test for uniform convergence] Let $f_n : X \to \mathbb{C}$ be a sequence of functions converging pointwise to a limit function f.*

1. *If $|f_n(x) - f(x)| \leq s_n$ for every $x \in X$, where $\{s_n\}_{n\in\mathbb{N}}$ is some sequence in $\mathbb{R}_{\geq 0}$ (independent of x) with $\lim_{n\to\infty} s_n = 0$, then f_n converge uniformly to f on X.*
2. *If there exists a sequence $x_n \in X$ such that $|f_n(x_n) - f(x_n)| \geq c$ for some positive constant c, then f_n does not converge uniformly to f on X.*

PROOF.

□

EXAMPLE. Show that the sequence of functions $f_n(z) = e^z + \frac{1}{n}$ converges to e^z uniformly on $\mathbb{C}$.

EXAMPLE. Show that for any $R > 0$ the sequence of functions $f_n(z) = e^z + \frac{z}{n}$ converges to e^z uniformly on $\{z \in \mathbb{C} \,:\, |z| < R\}$. Does is converge uniformly on $\mathbb{C}$?

THEOREM 5.5. *[Weierstrass M-test] Let $f_n : X \to \mathbb{C}$ be a sequence of functions such that $\left|f_n(x)\right| \leq M_n$ for all $x \in X$ and some sequence of non-negative numbers $\{M_n\}_{n\in\mathbb{N}}$ such that*

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Then $S_N(x) = \sum_{n=1}^{N} f_n(x)$ converges uniformly on X to some limit function $S : X \to \mathbb{C}$ which we denote by

$$S(x) = \sum_{n=1}^{\infty} f_n(x).$$

In particular, if all the functions $f_n(x)$ are continuous on X then $S(x) = \sum_{n=1}^{\infty} f_n(x)$ is also continuous on X.

PROOF.

□

EXAMPLE. Show that

$$\sum_{n=1}^{\infty} \frac{|2z|^{3n}}{3^{2n} n^2}$$

converges uniformly to a continuous function on $\mathbb{D}$.

THEOREM 5.6. *Assume a sequence of functions $f_n : [a,b] \to \mathbb{R}$ converge uniformly on an interval $[a,b]$ to some function f, and that $\{f_n\}_{n\in\mathbb{N}}$ are all continuous. Then for any $c \in [a,b]$ we have that:*

$$\lim_{n\to\infty} \int_a^c f_n(x)\,dx = \int_a^c f(x)\,dx.$$

In particular, if $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on an interval $[a,b]$ and if $\{f_n\}_{n\in\mathbb{N}}$ are continuous for all $n \in \mathbb{N}$ then for any $c \in [a,b]$ we have that:

$$\int_a^c \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_a^c f_n(x)\,dx.$$

5.1. Locally uniform convergence

DEFINITION 5.7 (Locally uniform convergence). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a metric space X. We say $\{f_n\}$ **converges locally uniformly (on X) to (the limit function) f**, if for every $x \in X$ there exists an open set $U_x \subset X$ (that can depend on x!) containing x on which $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f.

EXAMPLE. Show that the sequence of functions $f_n(z) = z^n$ converges locally uniformly on $\mathbb{D}$ but not uniformly.

THEOREM 5.8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions, which converges locally uniformly on X to a limit function f. Then f is continuous on X.*

PROOF.

$\square$

THEOREM 5.9. *[Local M-test] Let X be a metric space and let $f_n : X \to \mathbb{C}$ be a sequence of continuous functions such that for any $x_0 \in X$, there is an open $U_{x_0} \subset X$ containing x_0 and constants $M_n\left(U_{x_0}\right) > 0$ (which may depend on U_{x_0}!) such that $|f_n(x)| \le M_n\left(U_{x_0}\right)$ for all $x \in U_{x_0}$, and $\sum_{n=1}^{\infty} M_n\left(U_{x_0}\right) < \infty$. Then $\sum_{n=1}^{\infty} f_n$ converges locally uniformly to a continuous function on X.*

PROOF.

$\square$

EXAMPLE. Show that the series $\sum_{n=1}^{\infty} \frac{\left(z+\frac{1}{z}\right)^n}{n!}$ converges locally uniformly to ao continuous function on $\mathbb{C}^*$.

5.2. Complex power series

DEFINITION 5.10. A (complex) power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n(z-c)^n$$

where $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers and $c \in \mathbb{C}$.

THEOREM 5.11. *For any sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}}$ we can define the power series*

$$S(z) = \sum_{n=0}^{\infty} a_n(z-c)^n.$$

There exists $R \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that

- *$S(z)$ converges only for $z = c$ when $R = 0$. In this case $S(c) = a_0$.*
- *$S(z)$ converges absolutely for all $|z-c| < R$ when $R > 0$. If $R = +\infty$ this condition holds for any z.*
- *$S(z)$ diverges for $|z-c| > R$ when $R > 0$. If $R = +\infty$ this condition never holds.*

*R is called the **radius of convergence** of our power series and $B_R(c)$ is called the **disc of convergence**.*

Recall/Fact: The radius of convergence can be found using the formula

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

We can replace $\limsup$ by $\lim$ when the limit exists.
Moreover, we have the following formula when the limit exist:

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

(note that the above implies that if both limits exist they are the same).

THEOREM 5.12. *A power series* $\sum_{n=0}^{\infty} a_n(z-c)^n$ *with radius of convergence* $0 < R \leq \infty$ *converges uniformly on any ball* $B_r(c)$ *with* $0 < r < R$. *This implies the power series is locally uniformly convergent on its disc of convergence.*

PROOF.

$\square$

EXAMPLE. Show that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges locally uniformly on $\mathbb{C}$.

Differentiation/integration of power series.

PROPOSITION 5.13. *[Term by term differentiation or integration preserves the radius of convergence] Let* $\sum_{n=0}^{\infty} a_n(z-c)^n$ *be a power series with radius of convergence* $0 < R \leq +\infty$. *Then the formal derivatives and anti-derivatives of the power series*

$$\sum_{n=1}^{\infty} n a_n (z-c)^{n-1} \qquad \textit{and} \qquad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$$

are power series with the same radius of convergence R.

THEOREM 5.14. *[Power series can be differentiated term by term in their disc of convergence] Let* $\sum_{n=0}^{\infty} a_n(z-c)^n$ *be a power series in* $\mathbb{C}$*, with radius of convergence* $0 < R \leq +\infty$*, and let* $f : B_R(c) \to \mathbb{C}$ *be the resulting limit function. Then* f *is holomorphic on* $B_R(c)$ *with*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-c)^{n-1}$$

for $z \in B_R(c)$.

PROOF.

□

COROLLARY 5.15. *A power series f of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$ with positive radius of convergence R can be differentiated infinitely many times in $B_R(c)$. We have that*

$$f^{(k)}(z) := \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\,a_n(z-c)^{n-k} = \sum_{n=k}^{\infty} k!\binom{n}{k} a_n(z-c)^{n-k}$$

for $z \in B_R(c)$ which implies that $f^{(k)}(c) = k!\,a_k$.

COROLLARY 5.16 (Power series can be integrated term by term in their disc of convergence). *A power series f of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$ with positive radius of convergence R has a holomorphic antiderivative $F : B_R(c) \to \mathbb{C}$, that is a holomorphic function F on $B_R(c)$ such that $F'(z) = f(z)$. F is given by $F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$ for $z \in B_R(c)$.*

CHAPTER 6

Complex integration over contours

6.1. Definition of contour integrals

DEFINITION 6.1. Consider a continuous function $f : [a,b] \to \mathbb{C}$ where $[a,b] \subset \mathbb{R}$. We define

$$\int_a^b f(t)\,dt = \int_a^b \big(\mathrm{Re}\big(f(t)\big) + i\mathrm{Im}\big(f(t)\big)\big)\,dt := \int_a^b \mathrm{Re}\big(f(t)\big)\,dt + i\int_a^b \mathrm{Im}\big(f(t)\big)\,dt.$$

EXAMPLE. Find $\int_0^1 f(t)dt$ where $f(t) = t + it$,

LEMMA 6.2.

1. *Let f_1 and f_2 be continuous functions from $[a, b]$ to $\mathbb{C}$. Then*
$$\int_a^b \big(f_1(t) + f_2(t)\big)\, dt = \int_a^b f_1(t)dt + \int_a^b f_2(t)dt.$$
2. *For any complex number $c \in \mathbb{C}$, and continuous function $f : [a, b] \to \mathbb{C}$,*
$$\int_a^b cf(t)dt = c\int_a^b f(t)dt.$$

PROOF.

$\square$

Recall: We say that a path $\gamma : [a,b] \to \mathbb{C}$ is C^1 if its real part and imaginary parts are continuously differentiable on $[a,b]$. At the end point a and b this means that real part and imaginary parts have right-sided derivatives at a and left-sided derivative at b, and that the derivatives are continuos from the right at a and from the left at b.
In that case we define

$$\gamma'(t) = \left(\mathrm{Re}\left(\gamma(t)\right)\right)' + i\left(\mathrm{Im}\left(\gamma(t)\right)\right)'.$$

DEFINITION 6.3 (Contours). Let $\gamma : [a,b] \to \mathbb{C}$ be a curve, and suppose that there exist $a = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = b$ such that the curves $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$, $i = 1,2,\ldots,n$ defined by $\gamma_i(t) := \gamma(t)$ for $t \in [a_{i-1}, a_i]$ are C^1 curves. Then we say that γ is a piecewise C^1-curve, or a **contour**.

DEFINITION 6.4. Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a continuous function. Let $\gamma : [a,b] \to U$ be a C^1-curve. Then we define the integral of f along the curve γ by

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If $\gamma : [a,b] \to U$ is a contour such that $\gamma|_{[a_{i-1},a_i]} \to \mathbb{C}$, with $a = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = b$ is a C^1 curves for $i = 1,\ldots,n$, then we define

$$\int_\gamma f(z)dz = \sum_{i=1}^n \int_{\gamma_i} f(z)dz.$$

LEMMA 6.5 (Basic properties). *Assuming that f_1, f_2 and f are continuous on U we find that for any C^1 curve $\gamma : [a,b] \to U$:*

1.

$$\int_\gamma (f_1(z) + f_2(z))dz = \int_\gamma f_1(z)dz + \int_\gamma f_2(z)dz.$$

2. For any $c \in \mathbb{C}$

$$\int_\gamma cf(z)dz = c\int_\gamma f(z)dz.$$

3. Defining $(-\gamma) : [-b,-a] \to U$ by $(-\gamma)(t) = \gamma(-t)$ we have that

$$\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz.$$

PROOF.

□

EXAMPLE. Consider the path $\gamma : [0,2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = re^{i\theta}$ with $r > 0$. Find:

- $\int_\gamma dz$.
- $\int_\gamma \overline{z}dz$.
- $\int_\gamma z^n dz$ for $n \in \mathbb{Z}$.

LEMMA 6.6 (Reparametrisation of curves). *Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be continuous, and let $\gamma : [a, b] \to U$ be a C^1 curve. If $\varphi : [a', b'] \to [a, b]$ is continuously differentiable bijection with $\varphi(a') = a$ and $\varphi(b') = b$ and we define $\delta : [a', b'] \to \mathbb{C}$ by*

$$\delta(t) := \gamma\big(\varphi(t)\big) = \big(\gamma \circ \varphi\big)(t)$$

then

$$\int_\gamma f(z)\,dz = \int_\delta f(z)\,dz.$$

PROOF.

□

Important notation: Given a domain D such that there exists a bijective contour $\gamma : [a,b] \to \partial D$ with a continuous inverse $\gamma^{-1} : \partial D \to [a,b]$ and such that $\gamma'(t) \neq 0$, we define

$$\int_{\partial D} f(z)dz = \int_{\gamma} f(z)dz.$$

This notion is well defined and doesn't depend on γ due to our Reparametrisation of curves lemma, Lemma 6.6. When the boundary has no "end points", such as the circle, we can extend to above definition to the case where $\gamma : [a,b) \to \partial D$ has the previously mentioned properties and $\gamma(a) = \gamma(b)$.
For example, for $D = \{z \in \mathbb{C} \,:\, |z-c| < r,\ r > 0, c \in \mathbb{C}\}$ we have that

$$\partial D = \{z \in \mathbb{C} \,:\, |z-c| = r,\ r > 0, c \in \mathbb{C}\},$$

which is the bijective image of the C^1 curve $\gamma : [0, 2\pi) \to \mathbb{C}$

$$\gamma_{c,r}(\theta) = c + re^{i\theta}.$$

Consequently

$$\int_{|z-c|=r} f(z)dz = \int_{\gamma_{c,r}} f(z)dz = \int_0^{2\pi} f\left(c + re^{i\theta}\right) rie^{i\theta} d\theta.$$

DEFINITION 6.7. Let if $\gamma : [a,b] \to \mathbb{C}$ and $\delta : [c,d] \to \mathbb{C}$ be two contours such that $\gamma(b) = \delta(c)$. We define their addition, $\gamma \cup \delta$, as the curve

$$\gamma \cup \delta : [a, b+d-c] \to \mathbb{C}$$

with

$$\gamma \cup \delta(t) := \begin{cases} \gamma(t), & , a \leq t \leq b, \\ \delta(t+c-b), & b \leq t \leq b+d-c. \end{cases}$$

By the definition of integration along a contour and the change of variables formula we have that

$$\int_{\gamma\cup\delta} f(z)dz = \int_\gamma f(z)dz + \int_\delta f(z)dz.$$

6.2. The Fundamental Theorem of Calculus

THEOREM 6.8 (**Complex Fundamental Theorem of Calculus - Part I (FTC-I)**). *Let $U \subset \mathbb{C}$ be an open set and let $F : U \to \mathbb{C}$ be holomorphic with continuous derivative f. Then for any contour $\gamma : [a,b] \to U$ we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular if γ is closed, that is $\gamma(a) = \gamma(b)$, then we have that

$$\int_\gamma f(z)dz = 0.$$

PROOF.

□

DEFINITION 6.9 (Length of a contour). Let $\gamma : [a,b] \to \mathbb{C}$ be a contour. We define the *length* of γ by

$$L(\gamma) := \int_a^b \left|\gamma'(t)\right| dt.$$

LEMMA 6.10 (The Estimation Lemma). *Let $f : U \to \mathbb{C}$ be continuous and $\gamma : [a,b] \to U$ be a contour. Then*

$$\left|\int_\gamma f(z)\,dz\right| \le \left(\sup_{z\in\gamma} |f(z)|\right) L(\gamma).$$

PROOF.

□

The definition of a length of a curve is motivated from the analysis of functions of many variables and the study of integration over curves. There one shows that the infinitesimal length of a curve is exactly $\left|\gamma'(t)\right| dt$. This means that if we have a function $g : U \to \mathbb{R}$ and a curve $\gamma : [a,b] \to U$ we can define the integral of the *real valued function* g along the curve γ as

$$\int_a^b g\left(\gamma(t)\right)\left|\gamma'(t)\right| dt.$$

We have a special notation for this:

Notation: Let $U \subseteq \mathbb{C}$ be a open and let $g : U \to \mathbb{R}$ be continuous. For any $\gamma : [a,b] \to U$ we define

$$\int_\gamma g(z)\, d\,|z| := \int_a^b g\left(\gamma(t)\right)\left|\gamma'(t)\right| dt.$$

In the proof of our estimation lemma we have shown the important inequality:

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma \left|f(z)\right| d\,|z|.$$

EXAMPLE. Consider $\gamma : \left[0, \frac{\pi}{2}\right] \to \mathbb{C}$ given by $\gamma(\theta) = 2e^{i\theta}$. Find an upper bound for

$$\left|\int_\gamma \frac{z+4}{z^3-1}\,dz\right|.$$

THEOREM 6.11 (**Complex Fundamental Theorem of Calculus - Part II (FTC-II)**). *Let $f : D \to \mathbb{C}$ be continuous on a domain D. If $\int_\gamma f(z)\,dz = 0$ for all closed contours γ in D, then there exists a holomorphic $F : D \to \mathbb{C}$ such that*

$$F'(z) = f(z).$$

PROOF.

$\square$