Complex Analysis II.
Prerequisites from Analysis I *

The definitions and results collected in this document serve as a refresher of some concepts from Analysis I that we will explore in the complex setting throughout the Complex Analysis II module. The definitions and results collected here are taken from the Analysis I lecture notes from 2021–2022 by Prof. D. Schütz and Prof. J. Funke.

Sequences.

Definition 1 (Real sequence)**.** *A real sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. That is, it assigns to every natural number $n \in \mathbb{N}$ a real number $x_n \in \mathbb{R}$. Such a sequence is denoted $\{x_n\}$.*

Definition 2 (Convergence of sequence)**.** *A real sequence $\{x_n\}$ is said to be convergent to the limit $x \in \mathbb{R}$ if*

$$\lim_{n\to\infty} |x_n - x| = 0.$$

That is, for every $\epsilon > 0$, there exists and index $N \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \quad \text{for all } n > N.$$

We write $\lim_{n\to\infty} x_n = x$, or we say "$x_n \to x$ as $n \to \infty$." A sequence that has a limit is called a convergent sequence. If a sequence is not convergent, then it is called divergent.

Definition 3 (Bounded sequence)**.** *Let $\{x_n\}$ be a real sequence and denote the set $X = \{x_n \in \mathbb{R} : n \in \mathbb{N}\}$. The sequence $\{x_n\}$ is called bounded above, respectively below, if X is bounded above, respectively below. The sequence $\{x_n\}$ is called bounded if X is bounded.*

Theorem 4 (COLT)**.** *Let $\{x_n\}$ and $\{y_n\}$ be real sequences that are convergent with limits $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Let $a, b \in \mathbb{R}$. Then we have*

1. $ax_n + by_n \to ax + by$ *as* $n \to \infty$.
2. $x_n \cdot y_n \to x \cdot y$ *as* $n \to \infty$.
3. $\frac{x_n}{y_n} \to \frac{x}{y}$ *as* $n \to \infty$.

Definition 5 (Subsequence)**.** *Let $\{x_n\}$ be a sequence. A subsequence of $\{x_n\}$ is a sequence $\{x_{n_j}\}$ with $n_1 < n_2 < n_3 < \cdots$.*

Theorem 6 (Bolzano–Weierstrass)**.** *Let $\{x_n\}$ be a bounded real sequence. Then $\{x_n\}$ has a subsequence which is convergent.*

Definition 7 (Lim sup and Lim inf)**.** *Let $\{x_n\}$ be a bounded sequence. The limit superior of $\{x_n\}$ is defined as*

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 1} \left\{ \sup_{m\geq n} \{x_m\} \right\},$$

and the limit inferior of $\{x_n\}$ is defined as

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 1} \left\{ \inf_{m\geq n} \{x_m\} \right\}.$$

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Series

Definition 8 (Series, convergence of series). *Let $\{a_n\}$ be a real sequence. Then the sequence of partial sums $\{s_k\}$, defined as*

$$s_k = \sum_{n=0}^{k} a_n = a_1 + a_2 + \cdots + a_k,$$

is called an (infinite) series. If the sequence of partial sums $\{s_k\}$ is convergent, then we say that the series $\sum_{n=0}^{\infty} a_n$ is convergent and we write

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} s_k.$$

Otherwise, we say that the series $\sum_{n=0}^{\infty} a_n$ is divergent.

Theorem 9 (COLT for series). *Assume the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge with limits a and b respectively. Let $c \in \mathbb{R}$. Then*

1. *$\sum_{n=0}^{\infty}(a_n + b_n)$ is convergent with limit $a + b$.*
2. *$\sum_{n=0}^{\infty} ca_n$ is convergent with limit ca.*

Theorem 10 (Comparison Test). *Let $N \in \mathbb{N}$, $\{a_n\}_{n\geq N}$ and $\{b_n\}_{n\geq N}$ be sequences with $0 \leq a_n \leq b_n$ for all $n \geq N$.*

1. *If $\sum_{n=0}^{\infty} b_n$ is convergent with limit b, then $\sum_{n=0}^{\infty} a_n$ is also convergent with limit $a \leq b$.*
2. *If $\sum_{n=0}^{\infty} a_n$ is divergent, then so is $\sum_{n=0}^{\infty} b_n$.*

Definition 11 (Absolute convergence). *We say that the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.*

Theorem 12 (Ratio Test). *Let $\{a_n\}$ be a sequence with $a_n \neq 0$ for all but possibly finitely many n.*

1. *If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.*
2. *If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then $\sum_{n=0}^{\infty} a_n$ is divergent.*

Theorem 13 (Root Test). *For a sequence $\{a_n\}$ set*

$$a = \limsup |a_n|^{1/n}.$$

1. *If $a < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.*
2. *If $a > 1$, then $\sum_{n=0}^{\infty} a_n$ is divergent.*

Functions, Limits, and Continuity.

Proposition 14 (Properties of image and preimage). *Let $f : X \to Y$ be a function, and assume that $A, B \subset X$. Then*

1. *$f(A \cap B) \subset f(A) \cap f(B)$.*
2. *$f(A \cup B) = f(A) \cup f(B)$.*
3. *$f(X \setminus A) \supset f(X) \setminus f(A)$.*

Assume that $C, D \subset Y$. Then

1. *$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.*
2. *$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.*

3. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$.

Definition 15 (Exponential function and Logarithm)**.** *The exponential function* $\exp : \mathbb{R} \to (0, \infty)$ *is defined as*

$$\exp(x) := e^x.$$

The logarithm function $\log : (0, \infty) \to \mathbb{R}$ *is defined for* $x > 0$ *by*

$$\log(x) = y,$$

where y *is the unique real number such that* $\exp y = x$.

- Let $a, b \in \mathbb{R}$, $a \leq b$. We call $(a, b) := \{t \in \mathbb{R} : a < t < b\}$ an open interval inside $\mathbb{R}$ (where we also allow $a = -\infty$ or $b = \infty$), and $[a, b] := \{t \in \mathbb{R} : a \leq t \leq b\}$ a closed interval or compact interval inside $\mathbb{R}$ (where we do not allow $a, b = \pm\infty$).
- We call a subset $X \subseteq \mathbb{R}$ open if for each $c \in X$ there exists $\epsilon > 0$ such that the open interval $(c - \epsilon, c + \epsilon) \subseteq X$.
- We call $c \in X$ and interior point if there exists an open subset U or an open interval (a, b) containing c which lies completely in U.

Throughout we let $f : X \to \mathbb{R}$ be a function on a subset $X \subseteq \mathbb{R}$.

Definition 16 (Continuous function)**.** *Let* $f : X \to \mathbb{R}$ *be a function and let* c *be an interior point of* X*. Then* f *is called continuous at* $c \in X$ *if*

$$\lim_{x \to c} f(x) = f(c).$$

That is, for all $\epsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$|f(x) - f(c)| < \epsilon \quad \text{for all } x \in X \text{ with } |x - c| < \delta.$$

The function $f : X \to \mathbb{R}$ *is called continuous if it is continuous for all* $c \in X$.

Proposition 17 (Continuity via sequences)**.** *Let* $X \subset \mathbb{R}$*,* $c \in X$*, and* $f : X \to \mathbb{R}$ *be a function. Then* f *is continuous at* c *if and only if for all sequences* $\{x_n\}$ *in* X *with* $x_n \to c$ *as* $n \to \infty$ *we have* $f(x_n) \to f(c)$ *as* $n \to \infty$*. That is,*

$$\lim_{n \to \infty} f(x_n) = f\big(\lim_{n \to \infty} x_n\big).$$

Theorem 18 (COLT for continuous functions)**.** *Let* $X \subset \mathbb{R}$*,* $c \in X$ *and* $f, g : X \to \mathbb{R}$ *be continuous functions at* c*. Then*

1. $a \cdot f(x) + b \cdot g(x)$ *is continuous at* c *for any* $a, b \in \mathbb{R}$.
2. $f(x) \cdot g(x)$ *is continuous at* c.
3. $f(x)/g(x)$ *is continuous at* c *provided that* $g(c) \neq 0$.

Theorem 19 (Composition of continuous functions is continuous)**.** *Let* $X, Y \subset \mathbb{R}$*,* $c \in X$*,* $f : X \to \mathbb{R}$*,* $g : Y \to \mathbb{R}$ *with* $f(X) \subset Y$*. If* f *is continuous at* $c \in X$ *and* g *is continuous at* $f(c) \in Y$*, then* $g \circ f$ *is continuous at* $c \in X$.

Theorem 20 (Extreme Value Theorem)**.** *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be continuous. Then* f *attains its maximum and minimum values on* $[a, b]$*. That is, every continuous function on a compact interval attains its maximum and minimum.*

Differentiable functions.

Definition 21 (Differentiable function)**.** *Let $X \subseteq \mathbb{R}$ be open and $f : X \to \mathbb{R}$ be a function. We say that f is differentiable at $c \in X$ if*

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

exists. We denote this limit $f'(c)$ and call $f'(c)$ the derivative of f at c. We call f a differentiable function if f is differentiable at all points $c \in X$. Another formulation of this is

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

Theorem 22 (Sums, products, compositions, quotients of differentiable functions)**.**

1. *Let $f, g : X \to \mathbb{R}$ be two functions which are differentiable at $c \in X$ and let $a \in \mathbb{R}$ be a constant. Then $f + g$ and af are also differentiable at c and*

$$(f + g)'(c) = f'(c) + g'(c).$$
$$(af)'(c) = af'(c).$$

Their product fg is also differentiable at c with

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

2. *Let $f, g : X \to \mathbb{R}$ be two functions such that g is differentiable at c and f is differentiable at $g(c)$. Then the composition $f \circ g$ is also differentiable at c and*

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

3. *Let $f : X \to \mathbb{R}$ be a function which is differentiable at c and such that $f(c) \neq 0$. Then $1/f$ is also differentiable at c and*

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f^2(c)}.$$

Power series.

Let $c \in \mathbb{R}$.

Definition 23 (Power series)**.** *A real power series is an infinite series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ with real a_n and $x \in \mathbb{R}$.*

Theorem 24 (Cauchy–Hadamard)**.** *Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ be a power series. Then there exists a constant $R \in [0, \infty]$ such that*

1. *If $R = 0$, then $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges only for $x = c$.*
2. *If $R > 0$, then*

$$\sum_{n=0}^{\infty} a_n(x-c)^n \text{ converges absolutely for } x \in (c - R, c + R);$$
$$\sum_{n=0}^{\infty} a_n(x-c)^n \text{ diverges for } |x - c| > R.$$

If we set

$$\limsup_{n} |a_n|^{1/n} = k \in [0, \infty],$$

then R is explicitly given by

$$R = \frac{1}{k} \in [0, \infty].$$

We call R the radius of convergence of the power series.

Lemma 25 (Term by term differentiation or integration preserves the radius of convergence)**.** *Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ be a power series with radius of convergence R. Then the*

$$\text{formal derivative } \sum_{n=0}^{\infty} na_n(x-c)^{n-1} = \frac{1}{x-c}\sum_{n=0}^{\infty} na_n(x-c)^n;$$
$$\text{formal antiderivative } \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} = (x-c)\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^n$$

also have radius of convergence R.

Theorem 26 (Power series can be differentiated term-by-term in their disc of convergence)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ be a power series and $R \in (0,\infty]$ be its radius of convergence. Then f is differentiable infinitely many times at all points $x \in (c-R, c+R)$, and we can differentiate term-by-term:*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}.$$

Sequences of functions, uniform convergence, and limit theorems.

Definition 27 (Pointwise convergence)**.** *Let $\{f_n\}$ be a sequence of functions on an interval I. We say that $\{f_n\}$ has a pointwise limit if for all $x \in I$, the limit $\lim_{n\to\infty} f_n(x)$ exists (as a sequence of real numbers). In that case, the limit function $f : I \to \mathbb{R}$ is defined as*

$$f(x) = \lim_{n\to\infty} f_n(x).$$

In other words, we have

$$\forall x \in I \quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N : \quad |f_n(x) - f(x)| < \epsilon.$$

Definition 28 (Uniform convergence)**.** *Let $\{f_n\}$ be a sequence of functions on an interval I. We say that $\{f_n\}$ converges uniformly to f if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in I$, we have*

$$|f_n(x) - f(x)| < \epsilon.$$

If f_n converges uniformly to f, we write "$f_n \to f$ uniformly". In other words, we have

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad \forall x \in I : \quad |f_n(x) - f(x)| < \epsilon.$$

Here, N does not depend on the individual point x, the same N works for all $x \in I$.

Theorem 29 (Uniform limits of continuous functions are continuous)**.** *Let f_n be a sequence of continuous functions on an interval I such that $f_n \to f$ uniformly. Then the limit function f is also continuous.*

Theorem 30 (Weierstrass M-test)**.** *Let $I \subset \mathbb{R}$ be an interval and $f_n : I \to \mathbb{R}$ be a sequence of functions such that*

$$|f_n(x)| \leq M_n \text{ for all } x \in I \quad \text{and} \quad \sum_{n=1}^{\infty} M_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly on } X \text{ to some limit function } f : X \to \mathbb{C}.$$

Integration.

Theorem 31 (Fundamental Theorem of Calculus)**.** *Let f be a continuous function on $[a,b]$. Then*

$$F(x) := \int_a^x f(t)\,dt$$

is a differentiable function on $[a,b]$ (one-sided at a and b), and we have $F'(x) = f(x)$ for all $x \in [a,b]$.

Theorem 32 (Integral of uniform limit of continuous functions is limit of integrals)**.** *Let $I = [a, b]$ and $\{f_n\}$ be a sequence of continuous functions on I such that $f_n \to f$ uniformly. Then*

$$\lim_{n\to\infty} \int_a^c f_n(x)\,dx = \int_a^c f(x)\,dx, \qquad \textit{for all } c \in [a, b].$$

Complex numbers.

The following facts are recalled from the lecture notes from Analysis I, 2021–2022.

- Addition of complex numbers is associative and commutative. Multiplication of complex numbers is associative and commutative.
- For $z = x + iy$, we call $\overline{z} := x - iy$ the complex conjugate of z.
- For $z = x + iy$, we call $|z| := \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ the modulus or absolute value of z.
- $|z| = 0$ if and only if $z = 0$.
- $|z \cdot w| = |z| \cdot |w|$.
- (Triangle Inequality) For $z_1, z_2 \in \mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$.