Q.1 *Let X be a set and let $d : X \times X \to \mathbb{R}$ be defined as*

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Show that d is a metric on X (which we call the discrete metric).
Show in addition that if X is a non-trivial vector space, the discrete metric can't be induced by a norm. In other words, show that there exists no norm on X, $\|\cdot\|$, such that

$$d(x,y) = \|x - y\|.$$

<u>Solution:</u>

- By def $d(x,y) \geq 0$ $\forall x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$.
- $d(x,y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases} = \begin{cases} 0 & y=x \\ 1 & y \neq x \end{cases} = d(y,x)$
- Triangle inequality: Let $x,y,z \in X$

$$\underbrace{d(x,z)}_{\geq 0} + \underbrace{d(y,z)}_{\geq 0} = \begin{cases} 0 & x=y=z \\ 1 & \text{otherwise} \\ 2 & x \neq z \text{ and } y \neq z \end{cases} = \begin{cases} 0 & x=y=z \\ \geq 1 & \text{otherwise} \end{cases}$$

$$\geq \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases} = d(x,y)$$

$\Rightarrow d$ is a metric.

d can't come from a norm since it doesn't scale.

If d came from a norm

$$d(x,0) = \|x\|$$

$$d(\lambda x, 0) = |\lambda| \|x\| \quad \forall \text{ scalar } \lambda$$

if $x \neq 0$, $\lambda \neq 0$ $\quad d(x,0) = 1$, $d(\lambda x, 0) = 1$

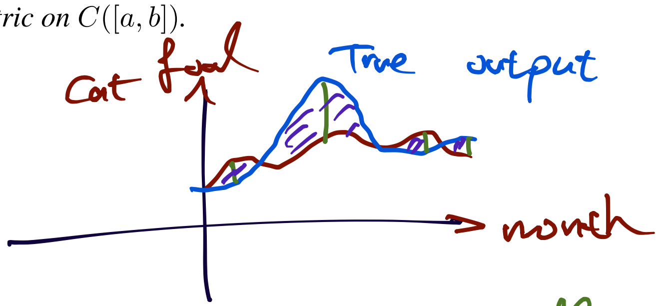
So $1 = |\lambda| 1$ not true for all λ!

Q.2 *(Assignment sheet 2 problem 3) In the space $C([a,b])$ of continuous functions defined on a closed interval $[a,b]$ (for $a < b$), let*

$$d_1(f,g) := \int_a^b |f(t) - g(t)|\, dt.$$

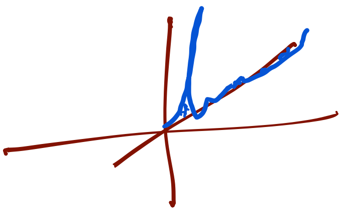
Show that d_1 is a metric on $C([a,b])$.

Background



$\max |f(x) - g(x)|$ ← worst difference between production and prediction

$d_1(f,g) = \int |f(t) - g(t)|\, dt$ ← "Total difference" over the year



Solution:

By def $|f(t) - g(t)| \geq 0 \quad \forall t$

$\Rightarrow d_1(f,g) = \int_a^b |f(t) - g(t)|\, dt \geq 0$

$d_1(f,g) = 0 \iff \int_a^b |f(t) - g(t)|\, dt = 0$

$h(t) = |f(t) - g(t)|$ is cont. since f, g and $|\cdot|$ are cont.

$h(t) \geq 0 \quad \forall t$

$\Rightarrow \int_a^b h(t)\, dt = 0 \iff h(t) = 0 \quad \forall t \in [a,b]$

$h(t) = 0 \iff f(t) = g(t) \quad \forall t$ i.e. $f \equiv g$.

Symmetry:

$$d_1(f,g) = \int_a^b |f(t)-g(t)|\,dt = \int_a^b |g(t)-f(t)|\,dt$$
$$= d_1(g,f).$$

Triangle inequlity

$$d_1(f,g) = \int_a^b |f(t)-g(t)|\,dt$$
$$= \int_a^b |f(t)-h(t)+h(t)-g(t)|\,dt$$

Triangle i.eq. for $|\cdot|$

$$\leq \int \left[|f(t)-h(t)| + |h(t)-g(t)| \right] dt$$
$$= \int_a^b |f(t)-h(t)|\,dt + \int_a^b |h(t)-g(t)|\,dt$$
$$= d_1(f,h) + d_1(g,h)$$

Remark: d_1 comes from a norm!

$$\|f\|_1 = \int_a^b |f(t)|\,dt$$

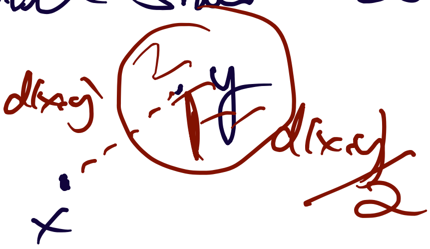
Q.3 *(Assignment sheet 2 problem 7)*

(i) *Show that in any metric space* (X, d) *the set* $\{x\}$*, consisting of a single point* $x \in X$*, is closed.*

(ii) *Show that in any metric space* (X, d) *the closed ball* $\overline{B}_r(x) := \{y \in X : d(y, x) \leq r\}$*, of radius* $r > 0$ *centred at* $x \in X$*, is closed.*

Solution:

(i) To show that $\{x\}$ is closed we will show that $\{x\}^c$ is open.



For any $y \neq x$ let $\varepsilon = \frac{d(x,y)}{2}$.

We have that $B_\varepsilon(y) \subseteq \{x\}^c$.

Indeed $x \notin B_\varepsilon(y)$ since

$$d(x,y) > \frac{d(x,y)}{2} = \varepsilon$$

(ii) Please read the solution.

Q.4 *(Assignment sheet 2 problem 13) Give an example of a metric space X and an $x \in X$ such that $\overline{B}_1(x) \neq \overline{B_1(x)}$; that is, the closure of the open ball is not necessarily the closed ball!!*

Consider X with the discrete metric. Let $x \in X$

$$B_1(x) = \{y \in X : d(x,y) < 1\} = \{x\}$$

$$\overline{B}_1(x) = \{y \in X : d(x,y) \leq 1\} = X$$

if we show that $\overline{\{x\}}$ we don't get X we found our example.

We will show that

$$\left(\left(\{x\}^c \right)^o \right)^c = \{x\}$$

Recall every set in discrete metric is open.

I claim that if A is open $A = A^o$

If that is true

$$\left(\underbrace{\left(\{x\}^c \right)^o}_{\text{open}} \right)^c = \left(\{x\}^c \right)^c = \{x\}$$

~~Q.4 *(Assignment sheet 2 problem 13) Give an example of a metric space X and an $x \in X$ such that $\overline{B}_1(x) \neq \overline{B_1(x)}$; that is, the closure of the open ball is not necessarily the closed ball!!*~~

<u>Idea:</u>

$$A^\circ \subseteq A$$

if A is open and $x \in A$

$\exists \varepsilon > 0$ s.t $B_\varepsilon(x) \subseteq A$

$\Rightarrow x \in A^\circ$.

x was arbitrary $\Rightarrow$ $A \subseteq A^\circ$

in conclusion $A = A^\circ$.