Q.1 *Let X be a set and let $d : X \times X \to \mathbb{R}$ be defined as*

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Show that d is a metric on X (which we call the discrete metric).
Show in addition that if X is a non-trivial vector space, the discrete metric can't be induced by a norm. In other words, show that there exists no norm on X, $\|\cdot\|$, such that

$$d(x,y) = \|x - y\|.$$

S.1 From its definition we have that $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$. Moreover

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases} = \begin{cases} 0, & y = x, \\ 1, & y \neq x, \end{cases} = d(y,x).$$

We are left with showing the triangle inequality. Let x, y and z be given. We have that

$$d(x,z) + d(z,y) = \begin{cases} 0, & x = y \text{ and } z = y, \\ 2, & x \neq z \text{ and } z \neq y, \\ 1, & \text{otherwise}, \end{cases} = \begin{cases} 0, & x = y = z, \\ \geq 1, & \text{otherwise}, \end{cases} \geq d(x,y)$$

which shows the first part of the problem.
To see that the distance is not induced by a norm we notice that if it was

$$d(x,0) = \|x\|$$

and consequently for any scalar λ

$$d(\lambda x, 0) = |\lambda| \, \|x\|.$$

If $x \neq 0$ then for any $\lambda \neq 0$ we have that $\lambda x \neq 0$ and as such $d(\lambda x, 0) = 1$. This implies that for any $x \neq 0$ and $\lambda \neq 0$ we have that

$$1 = |\lambda| \, \|x\|$$

which is impossible. Hence, the distance is not induced by a norm.

Q.2 *(Assignment sheet 2 problem 3) In the space $C([a,b])$ of continuous functions defined on a closed interval $[a,b]$ (for $a < b$), let*

$$d_1(f,g) := \int_a^b |f(t) - g(t)| \, dt.$$

Show that d_1 is a metric on $C([a,b])$.

S.2 We start by noticing that $|f(t) - g(t)| \geq 0$ and as such $\int_a^b |f(t) - g(t)| \, dt \geq \int_a^b 0 \, dt = 0$. Moreover, since both f and g are continuous, so if $|f - g|$ and

$$0 = d_1(f,g) = \int_a^b |f(t) - g(t)| \, dt.$$

implies, due to the continuity and non-negativity, that $|f(t) - g(t)| = 0$ for all $t \in [a,b]$ or that $f \equiv g$.
Since $|f - g| = |g - f|$ we have that $d_1(f,g) = d_1(g,f)$ and since for any f, g and h we have that

$$|f(t) - g(t)| = |(f(t) - h(t)) + (h(t) - g(t))| \leq |f(t) - h(t)| + |h(t) - g(t)|$$

we conclude that

$$d_1(f,g) \leq d_1(f,h) + d_1(h,g)$$

which is the desired triangle inequality.

Q.3 *(Assignment sheet 2 problem 7)*

(i) Show that in any metric space (X,d) the set $\{x\}$, consisting of a single point $x \in X$, is closed.

(ii) Show that in any metric space (X,d) the closed ball $\overline{B}_r(x) := \{y \in X : d(y,x) \leq r\}$, of radius $r > 0$ centred at $x \in X$, is closed.

S.3 (i) We need to show that the complement of $\{x\}$ is open. Take $y \in \{x\}^c$, i.e. $y \neq x$, and let $\epsilon = \frac{d(x,y)}{2}$. We claim that $B_\epsilon(y) \subseteq \{x\}^c$. Indeed, if $x \in B_\epsilon(y)$ then

$$d(x,y) < \epsilon = \frac{d(x,y)}{2}$$

which is impossible.

(ii) We need to show that complement of $\overline{B}_r(x)$ is open. By definition

$$\overline{B}_r(x)^c = \{y \in X \mid d(x,y) \leq r\}^c = \{y \in X \mid d(x,y) > r\}.$$

Let $y \in \overline{B}_r(x)^c$. We know that $d(x,y) > r$ and can define

$$\epsilon = \frac{d(x,y) - r}{2} > 0.$$

We claim that $B_\epsilon(y) \subseteq \overline{B}_r(x)^c$. Indeed, assume that $B_\epsilon(y) \cap \overline{B}_r(x) \neq \emptyset$. If $z \in B_\epsilon(y) \cap \overline{B}_r(x)$ then

$$d(x,y) \leq d(x,z) + d(y,z) \leq r + d(y,z) < r + \epsilon = \frac{r + d(x,y)}{2}.$$

This implies that $d(x,y) < r$ which is impossible.

Alternative proof: By the triangle inequality we know that for any x, y and z we have that

$$d(x,z) \leq d(x,y) + d(y,z)$$

and

$$d(y,z) \leq d(x,y) + d(x,z).$$

Combining the two inequalities we find that

$$d(x,y) \geq \max\{d(x,z) - d(y,z), d(y,z) - d(x,z)\} = |d(x,z) - d(y,z)|.$$

Thus, with $\epsilon = \frac{d(x,y)-r}{2}$, we see that if $z \in B_\epsilon(y)$

$$d(x,z) \geq d(x,y) - d(y,z) > d(x,y) - \epsilon = \frac{r + d(x,y)}{2} > r$$

showing that $B_\epsilon(y) \subseteq \overline{B}_r(x)^c$.

Q.4 *(Assignment sheet 2 problem 13) Give an example of a metric space X and an $x \in X$ such that $\overline{B}_1(x) \neq \overline{B_1(x)}$; that is, the closure of the open ball is not necessarily the closed ball!!*

S.4 Consider any set X that is not a singleton with the discrete metric and let $x \in X$ be arbitrary. We notice that

$$\overline{B}_r(x) = \{y \in X \ : \ d(x,y) \leq r\} = \begin{cases} \{x\}, & r < 1, \\ X, & r \geq 1 \end{cases}$$

Thus $B_1(x) = X$. On the other hand, as we saw in class

$$B_r(x) = \{y \in X \ : \ d(x,y) < r\} = \begin{cases} \{x\}, & r \leq 1, \\ X, & r > 1 \end{cases}$$

which implies that $B_1(x) = \{x\}$. We claim that $\overline{B_1(x)} = \{x\} \neq X$ which will give us the desired example. We have several way to show this but we will go with the original definition: $\overline{A} = \left((A^c)^0\right)^c$. Recall that with the discrete metric every set is open. We claim that for any open set A we have that $A = A^0$. Once this is proven we'll find that since every set is open in the discrete metric, we have that

$$\overline{A} = \left((A^c)^0\right)^c = (A^c)^c = A$$

which shows the desired result.
Proof of the claim By definition, $A^0 \subseteq A$ so we only need to show that $A \subseteq A^0$. Let $x \in A$. Since A is open we can find an open ball of radius ϵ for some $\epsilon > 0$ that is centred in x and is contained in A. In other words, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. By the definition of A^0 and the fact that pen balls are open sets we conclude that $x \in A^0$. As x was arbitrary we find that $A \subseteq A^0$ and the proof is complete.
Remark: If we use the theorem that states that A is closed if and only if $A = \overline{A}$ we can conclude that any set is in the discrete metric satisfies $A = \overline{A}$ since every set is closed.