Q.1 *Show the following statement from class: Let (X,d) be a metric space and let $A \subseteq X$. Then A is closed if and only if any sequence of elements of A that converges has its limit in A. In other words if $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ then $x \in A$.*

S.1 Let's start by assuming that A is closed and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of elements in A that converges to an element $x \in X$. If $x \notin A$ then $x \in A^c$. Since A is closed A^c is open and since $x \in A^c$ we can find $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq A^c$. n other words $B_{\epsilon_0}(x) \cap A = \emptyset$. This is impossible since we know that due to the convergence of $\{x_n\}_{n\in\mathbb{N}}$ we can find $N \in \mathbb{N}$ such that for all $n > N$ we have that $x_n \in B_{\epsilon_0}(x)$ but also $x_n \in A$.

Conversely, let us assume that any limit of any sequence of elements in A remains in A and show that A is closed. It is sufficient to show that A^c is open. Consider $x \in A^c$ and assume that there is no $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A^c$. In other words, for any $\epsilon > 0$ we can find $x_\epsilon \in A$ such that $x_\epsilon \in B_\epsilon(x)$. This means that for any $n \in \mathbb{N}$ we have $x_n \in A$ such that $x_n \in B_{1/n}(x)$. By definition this means that $d(x_n, x) < \frac{1}{n}$ which implies that $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Since $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of elements from A we conclude by our assumption that $x \in A$ which is a contradiction.

Q.2 *(i) (Assignment sheet 2 problem 15) Show that if a sequence $\{x_n\}$ converges in a discrete metric space, then it is eventually constant.*

(ii) Find a metric space (X,d) and a set $A \subseteq X$ such that A is closed and bounded but is not sequentially compact.

S.2 (i) Assuming that $x_n \to x_0$, we conclude that there exists $N \in \mathbb{N}$ such that for all $n > N$ we have that $x_n \in B_{\frac{1}{2}}(x)$. We saw in the discrete metric $B_{\frac{1}{2}}(x) = \{x\}$ and as such for all $n > N$ we have that $x_n = x$ which shows the desired result.

(ii) Consider the set $X = [0,1]$ we the discrete metric. X is bounded since $X \subseteq B_2(0)$ and is closed since it is the entire space (in fact every set is closed in the discrete metric). Consider the sequence $x_n = \frac{1}{n}$. The sequence is in X but since $x_n \neq x_m$ for any $n \neq m$ we see that it has no subsequence that is eventually constant. Consequently it has no subsequence that converges in the discrete metric so the set is not sequentially compact.

Q.3 *(Assignment sheet 4 problem 4) The real axis and the imaginary axis divide $\mathbb{C}$ into four quadrants as follows:*

$$\begin{array}{c|c} \Omega_2 & \Omega_1 \\ \hline \Omega_3 & \Omega_4 \end{array}$$

Determine the images of $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ under the exponential map $z \mapsto e^z$.

S.3 We have that $e^z = e^x e^{iy}$ so we know that segment of length more than 2π of the line $x = c$ goes into a circle of radius e^c. Consequently e^z takes Ω_1 and Ω_4 to the set

$$\left\{w \in \mathbb{C} \,:\, e^0 < |w| < e^\infty\right\} = \{w \in \mathbb{C} \,:\, |w| > 1\}$$

and it takes Ω_2 and Ω_3 to the set

$$\left\{w \in \mathbb{C} \,:\, 0 < |w| < e^0\right\} = \{w \in \mathbb{C} \,:\, 0 < |w| < 1\}.$$

Q.4 *(Assignment sheet 4 problem 6(b)) Give examples to illustrate that, in general, for complex numbers z, w,*

$$Log(zw) \neq Log(z) + Log(w).$$

S.4 By definition

$$Log(\xi) = \log(|\xi|) + iArg(\xi).$$

As such

$$Log(zw) = \log(|zw|) + iArg(zw) = \log(|z|) + \log(|w|) + iArg(zw)$$

and

$$Log(z) + Log(w) = \log(|z|) + iArg(z) + \log(|w|) + iArg(w).$$

This means that

$$Log(zw) \neq Log(z) + Log(w)$$

if and only if

$$Arg(zw) = Arg(z) + Arg(w)$$

which we know is not true. For instance, for $z = w = e^{-i\frac{\pi}{2}} = -i$ we have that

$$Arg(z) = Arg(w) = -\frac{\pi}{2}$$

but

$$Arg(zw) = Arg(-1) = \pi.$$