Q.1 *(Assignment sheet 4 problem 17) Let $f(z)$ be a holomorphic function. Prove the following variants of the* **Zero derivative theorem**, *which says that, if any one of the following conditions hold on a domain D then $f(z)$ is constant on D.*
(i) $f(z)$ is a real number for all $z \in D$.
(ii) the real part of $f(z)$ is constant on D.
(iii) the modulus of $f(z)$ is constant on D.

Solution:

(i) $f = u + iv$. From the assumption

$$v \equiv 0$$

Since f is holo.

$$u_x = v_y \qquad u_y = -v_x \qquad \forall x, y \in D$$

$$v_y = v_x = 0 \implies u_x = u_y = 0$$

$$\implies u = \text{Const}$$

(ii) $f = u + iv$. From the assumption

$$u = c_1 \text{ constant}$$

since f is holo.

$$u_x = v_y \qquad u_y = -v_x$$

$$u_x = u_y = 0 \implies v_x = v_y = 0$$

$$\implies v = c_2 \text{ constant}$$

$$\implies f = c_1 + ic_2$$

(iii) $|f(z)| = c$ for some $c \geq 0$. If $f = u + iv$

$$u^2 + v^2 = c^2$$

Implicit diff shows that

$$\begin{aligned} 2uu_x + 2vv_x &= 0 \\ 2uu_y + 2vv_y &= 0 \end{aligned} \iff \underbrace{\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}}_{A(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (*)$$

$$\det A(x,y) = \det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = u_x v_y - u_y v_x = u_x^2 + u_y^2$$

if $u(x_0, y_0) = v(x_0, y_0) = 0 \implies$

$$u^2(x_0, y_0) + v(x_0, y_0)^2 = 0 = c^2$$

$\Rightarrow c=0$ and then

$$u^2+v^2=0 \quad \forall x,y$$

i.e $u\equiv 0\equiv v$

Assume $c\neq 0$ then $u(x,y)$ and $v(x,y)$ can't be zero together.

$\Rightarrow$ (*) always has a non-trivial solution

$\Rightarrow$ $\det A(x,y)=0$ i.e

$$u_x^2+u_y^2=0$$

$\Rightarrow$ $u_x=u_y=0$ i.e $u=\text{const}$.

and we conclude by our previous sub-question.

Q.2 *(Assignment sheet 5 problem 5) In what subset of the complex plane is* $\sinh z$ *conformal? For every point* z_0 *at which the function is not conformal, give an example of two paths (lines) through* z_0 *such that the angle (or the orientation of the angle) between them is not preserved by* $f(z)$ *at* z_0.

Solution: $f(z) = \sinh z$ is entire

$\Rightarrow f(z)$ is holo in any point s.t

$f'(z) \neq 0$ and vice versa.

$$f'(z) = \cosh z = \frac{e^z + e^{-z}}{2}$$

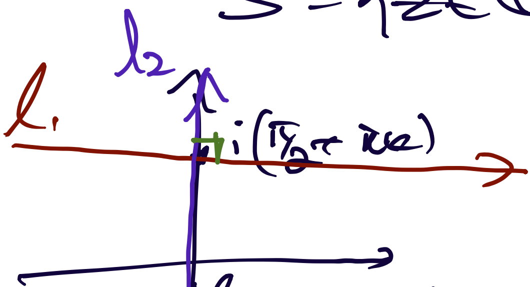
$$f'(z) = 0 \iff e^z = -e^{-z} \qquad (\cdot\, e^z)$$

$$\iff e^{2z} = -1 = e^{i\pi}$$

$$\iff 2z = i\pi + 2\pi i k \qquad k \in \mathbb{Z}$$

i.e. f is not conformal only on

$$S = \{ z \in \mathbb{C} \mid z = i(\pi/2 + \pi k) \quad k \in \mathbb{Z} \}$$



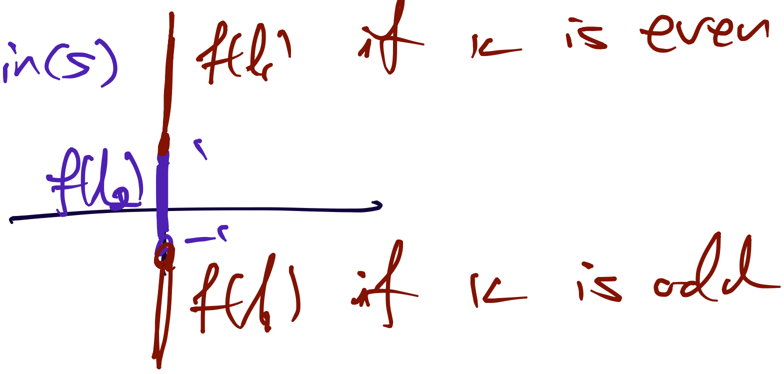
$l_1 :\ x = t \quad y = \pi/2 + \pi k \quad t \in \mathbb{R}$

$l_2 :\ x = 0 \quad y = s \quad s \in \mathbb{R}$

Recall $\sinh z = \sinh x \cos y + i \cosh x \sin y$

$f(l_1) = 0 + i \cosh t \, (-1)^k$

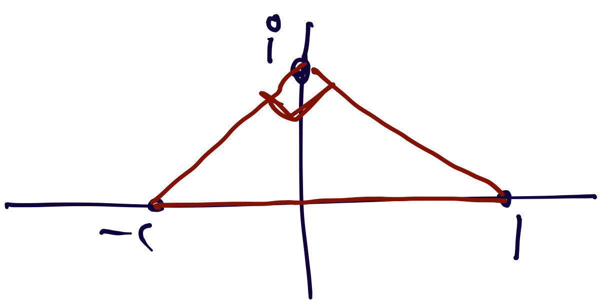
$f(l_2) = 0 + i \sin(s)$



The angle between the curves is π.

Q.3 *(Assignment sheet 6 problem 5) Is there a Möbius transformation which maps the sides of the triangle with vertices at $-1, i$ and 1 to the sides of an equilateral triangle (all sides of equal length)? Either give an example of such a Möbius transformation, or explain why it is not possible.*

Solution:



There is **no** such Möbius trans. since Möbius trans. are conformal on $\mathbb{C} \setminus \{-\frac{d}{c}\}$ and the angles between the sides of equilateral triangle are $\pi/3, \pi/3, \pi/3$ and not $\pi/4, \pi/2, \pi/4$.

Q.4 *(Assignment sheet 6 problem 8) If α and β are the two fixed points of a Möbius transformation $f(z)$, show that for all $z \neq \alpha, \beta$ and $f(z) \neq \infty$, we have*

$$\frac{f(z)-\alpha}{f(z)-\beta} = K\frac{z-\alpha}{z-\beta},$$

where K is a constant.

(see solution)

Q.5 *(Assignment sheet 6 problem 9) Find the Möbius transformation taking the ordered set of points $\{-i, -1, i\}$ to the ordered set of points $\{-i, 0, i\}$. What is the image of the unit disc under this map? Which point is sent to ∞?*

Solution: Using the cross ratio we find that for any z

$$(z, -i; -1, i) = (f(z), -i; 0, i)$$

$$\frac{z-(-1)}{z-i} \cdot \frac{\cancel{-i}-i}{-i-(-1)} = \frac{f(z)-0}{f(z)-i} \cdot \frac{\cancel{-i}-i}{-i-0}$$

$$\frac{f(z)}{f(z)-i} = -i\,\frac{z+1}{z-i} \cdot \frac{1}{1-i}$$

$$f(z)\underbrace{(1-i)(z-i)}_{z-iz-i-1} = -i\,(f(z)-i)\,(z+1)$$

$$f(z)\left(z(1-i)-i-1\right) = (f(z)-i)\,(-iz-i)$$

$$f(z)\left(\underbrace{z(1-i)}_{z-\cancel{iz}} \cancel{-i} -1 \cancel{+iz} \cancel{+i}\right) = -i\,(-iz-i)$$

$$f(z)\,(z-1) = -z-1$$

$$f(z) = -\frac{z+1}{z-1}$$

$$-\left(\frac{-i+1}{-i-1}\right) = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = -i$$

f takes takes $z=1$ to ∞.

$z=1$ is on $|z|=1 \longrightarrow$ the image of $|z|=1$ under f is a line.

... (see solution)