Q.1 *(Assignment sheet 4 problem 17) Let $f(z)$ be a holomorphic function. Prove the following variants of the* **Zero derivative theorem***, which says that, if any one of the following conditions hold on a domain D then $f(z)$ is constant on D.*
(i) $f(z)$ is a real number for all $z \in D$.
(ii) the real part of $f(z)$ is constant on D.
(iii) the modulus of $f(z)$ is constant on D.

S.1 (i) In this setting we have that $f(x+iy) = u(x,y)+iv(x,y)$ and $v \equiv 0$. As such, using the Cauchy-Riemann equations we find that

$$u_x = 0 = u_y$$

which implies that $u \equiv const$ and consequently $f \equiv const$.
(ii) In this setting we have that $f(x+iy) = u(x,y) + iv(x,y)$ and $u \equiv const_1$. As such, using the Cauchy-Riemann equations we find that

$$v_x = 0 = v_y$$

which implies that $v \equiv const_2$ and as such $f \equiv const$.
(iii) In this setting we have that $f(x+iy) = u(x,y) + iv(x,y)$ and

$$u^2 + v^2 = c$$

for some constant $c \geq 0$. If $c = 0$ then we must have $u = v \equiv 0$ and the result is concluded. We assume, thus, that $c > 0$ and notice that it implies that we can't have a point $x_0 + iy_0 \in D$ such that $u(x_0, y_0) = v(x_0, y_0) = 0$ (else $c = 0$). Moving towards using the Cauchy-Riemann equations we differentiate the above with respect to x and y to find that

$$uu_x + vv_x = 0,$$

and

$$uu_y + vv_y = 0.$$

The above can be rewriten as the system

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $u_x = v_y$ and $u_y = -v_x$ we find that the matrix $A(x,y) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$ satisfies

$$\det A(x,y) = u_x v_y - v_x u_y = u_x^2 + u_y^2.$$

As our system has a non-trivial solution for every x, y (since $c > 0$) we must have that

$$0 = \det A(x,y) = u_x^2 + u_y^2$$

for all x, y in the domain. Consequently $u_x = u_y = 0$ in the domain and as such $u \equiv const$. We can use point (ii) or redo the proof for v to conclude the result.

Q.2 *(Assignment sheet 5 problem 5) In what subset of the complex plane is $\sinh z$ conformal? For every point z_0 at which the function is not conformal, give an example of two paths (lines) through z_0 such that the angle (or the orientation of the angle) between them is not preserved by $f(z)$ at z_0.*

S.2 Since f is an entire function the points where f is conformal are the points where $f'(z) \neq 0$. We have that $0 = f'(z) = \cosh(z) = \frac{e^z + e^{-z}}{2}$ if and only if $e^z = -e^{-z}$ or alternatively $e^{2z} = -1 = e^{i\pi}$. We conclude that

$$2z = i\pi + 2\pi i k$$

or

$$z = i\left(\frac{\pi}{2} + \pi k\right)$$

for some $k \in \mathbb{Z}$. We conclude that the points where f is not conformal are points in the set

$$S = \left\{ z \in \mathbb{C} \ : \ z = i\left(\frac{\pi}{2} + \pi k\right) \text{ for some } k \in \mathbb{Z} \right\}.$$

For a fixed k consider the orthogonal lines

$$\ell_1 : \quad x = t, \ y = \frac{\pi}{2} + \pi k, \quad t \in \mathbb{R},$$

$$\ell_2 : \quad x = 0, \ y = s, \quad s \in \mathbb{R}.$$

The lines meet at $i\left(\frac{\pi}{2} + \pi k\right)$ which corresponds to $t = 0$ and $s = \frac{\pi}{2} + \pi k$. Using the fact that

$$f(x+iy) = \sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$

The image of these curves is given by

$$f(\ell_1) : \quad u = 0, v = (-1)^k \cosh(t)$$

and

$$f(\ell_2) : \quad u = 0, v = \sin(s)$$

$f(\ell_1)$ is the set $\{w \in \mathbb{C} \ : \ \operatorname{Im} w \geq 1\}$ when k is even and $\{w \in \mathbb{C} \ : \ \operatorname{Im} w \leq -1\}$ when k is odd, while $f(\ell_2)$ is the set $\{w \in \mathbb{C} \ : \ -1 \leq \operatorname{Im} w \leq 1\}$.
The point of intersection is

$$f\left(i\left(\frac{\pi}{2} + \pi k\right)\right) = \sinh\left(i\left(\frac{\pi}{2} + \pi k\right)\right) = -i\sin\left(i \cdot i\left(\frac{\pi}{2} + \pi k\right)\right) = i(-1)^k$$

and the angle between the images of the curves is π.

Q.3 *(Assignment sheet 6 problem 5) Is there a Möbius transformation which maps the sides of the triangle with vertices at $-1, i$ and 1 to the sides of an equilateral triangle (all sides of equal length)? Either give an example of such a Möbius transformation, or explain why it is not possible.*

S.3 There is no such Möbius transformation. One way to show this is to try and construct such transformation using the cross ration but a more geometric solution would be to recall that Möbius transformations are conformal at any point they are defined in (on $\mathbb{C}$) and that they take a boundary of a set to the boundary of its image - in our case the given triangle needs to go to the equilateral one. Consequently, the angle between the images of the lines connecting -1 and i and 1 and i must be perpendicular to one another which is impossible.

Q.4 *(Assignment sheet 6 problem 8) If α and β are the two fixed points of a Möbius transformation $f(z)$, show that for all $z \neq \alpha, \beta$ and $f(z) \neq \infty$, we have*

$$\frac{f(z) - \alpha}{f(z) - \beta} = K\frac{z - \alpha}{z - \beta},$$

where K is a constant.

S.4 We know that $f(\alpha) = \alpha$ and $f(\beta) = \beta$. Let $z_0 \neq \alpha, \beta$ be such that $w_0 = f(z_0) \neq \infty$ (also $w_0 \neq \alpha, \beta$ since Möbius maps are injective). Using the invariance of Möbius transformations under the cross ratio we conclude that for any $z \in \mathbb{C}$

$$(f(z), w_0; \alpha, \beta) = (z, z_0; \alpha, \beta).$$

Equivalently

$$\frac{f(z) - \alpha}{f(z) - \beta}\,\frac{w_0 - \beta}{w_0 - \alpha} = \frac{z - \alpha}{z - \beta}\,\frac{z_0 - \beta}{z_0 - \alpha}.$$

Denoting by

$$K = \frac{z_0 - \beta}{z_0 - \alpha}\,\frac{w_0 - \alpha}{w_0 - \beta}$$

gives us the desired result.

Q.5 *(Assignment sheet 6 problem 9) Find the Möbius transformation taking the ordered set of points* $\{-i, -1, i\}$ *to the ordered set of points* $\{-i, 0, i\}$. *What is the image of the unit disc under this map? Which point is sent to* ∞*?*

S.5 Using the cross ratio we find that

$$\frac{f(z)-0}{f(z)-i}\,\frac{-i-i}{-i-0} = \frac{z-(-1)}{z-i}\,\frac{-i-i}{-i-(-1)}$$

which implies that

$$\frac{f(z)}{f(z)-i} = -\frac{i}{1-i}\,\frac{z+1}{z-i}$$

or

$$f(z)\,(1-i)\,(z-i) = -\,(f(z)-i)\,i\,(z+1)\,.$$

This can be rewritten as

$$f(z)\,[(1-i)\,(z-i)+i\,(z+1)] = -\,(z+1)$$

or equivalently

$$f(z) = -\frac{z+1}{z-1}.$$

The image of the unit circle is a circline. As the point which is sent to infinity by f, $z=1$, is on the circle we conclude that the image of the unit circle is a line. Since the boundary of the unit disc is sent to the boundary of its image under f and since $f(0)=1$ we conclude that the image of the unit disc under the map is the right hand-side plane $\mathbb{H}_R$.