Q.1 *(Assignment sheet 7 problem 4) Find a Möbius transformation f from the upper half-plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$ that takes $1+i$ to 0 and (when considered as a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) also takes 1 to $-i$. Give an explicit formula for $f(z)$.*

Solution: We recall that the Cayley map $\mathcal{M}_C(z) = \frac{z-i}{z+i}$ (i.e $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$) takes $\mathbb{H}$ to $\mathbb{D}$.

$\Rightarrow$ The map $\underbrace{(\mathcal{M}_C)^{-1}}_{\mathcal{M}_{C^{-1}}} \circ f$ is a Möbius map that takes $\mathbb{H}$ to $\mathbb{H}$.

By H∂H theorem

$$\mathcal{M}_{C^{-1}} \circ f = \mathcal{M}_S \qquad S \in SL_2(\mathbb{R})$$

$\Rightarrow$ ($\mathcal{M}_C \circ \mathcal{M}_{C^{-1}} = Id$)

$$f = \mathcal{M}_C \circ \mathcal{M}_S = \mathcal{M}_{CS}$$

if $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$ then we can write $f = \mathcal{M}_T$ with

$$T = CS = \begin{pmatrix} a-ic & b-id \\ a+ic & b+id \end{pmatrix}$$

$$\Rightarrow \quad f(z) = \frac{(a-ic)z + (b-id)}{(a+ic)z + (b+id)}$$

$f(1+i) = 0 \Rightarrow (a-ic)(1+i) + (b-id) = 0$

$f(1) = -i \Rightarrow (a-ic) + (b-id) = -i\big((a+ic) + (b+id)\big)$

we also have $ad-bc=1$.

Solving this gives us

$$d = \pm 2\sqrt{2}, \quad b = 0, \quad a = \pm\frac{\sqrt{2}}{2}, \quad c = \mp\frac{\sqrt{2}}{2}$$

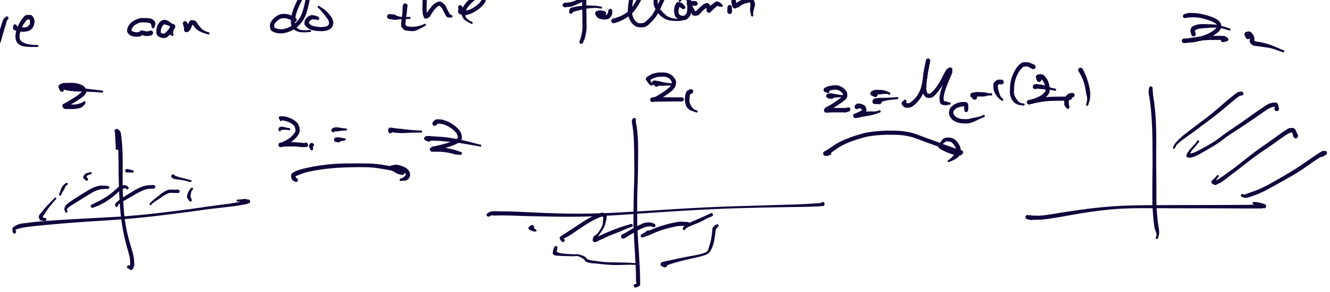
plugging it back and taking a common denominator gives

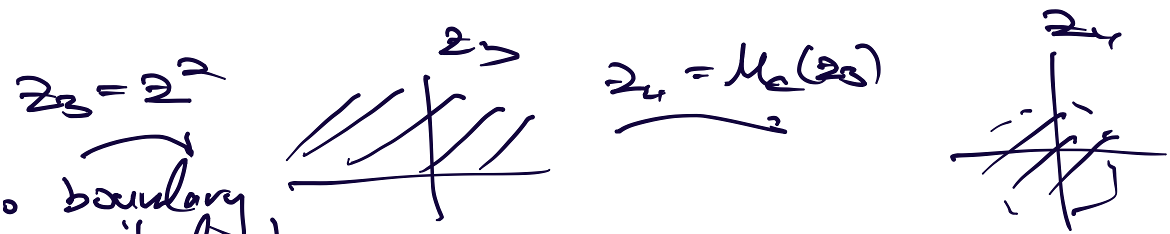
$$f(z) = \frac{(1+i)z - 2i}{(1-i)z + 2i}.$$

Q.2 *(Assignment sheet 7 problem 8) Use standard examples to find a biholomorphic map from the upper half $\Omega := \{z \in \mathbb{D} : Im(z) > 0\}$ of the unit disc onto the unit disc $\mathbb{D}$.*

Solution: We showed in class that the Cayley map M_c takes $\Omega_1 = \{z \in \mathbb{C} \mid Im z > 0, Re z > 0\}$ to the lower half disc

we can do the followin





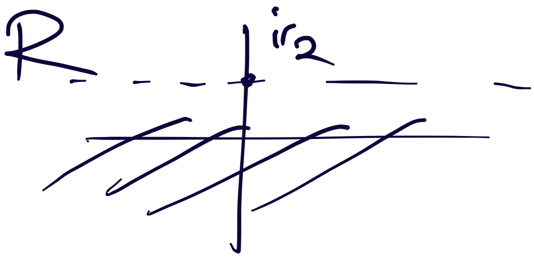
(no boundary ⇒ biholo) in this case

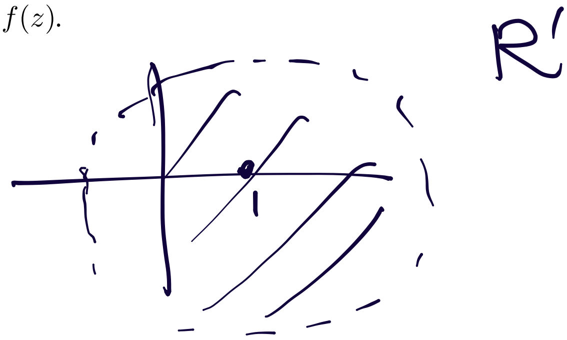
a biholo map taking the upper unit disc to $\mathbb{D}$ is given by

$$f(z) = M_c\left(M_c^{-1}(-z)\right)^2$$

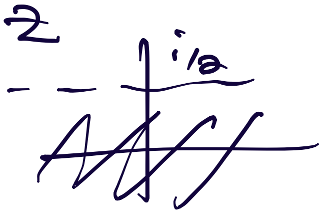
$$= \frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2}$$

Q.3 *(Assignment sheet 7 problem 11) Construct a biholomorphic map f from $\mathcal{R}$ onto $\mathcal{R}'$, where $\mathcal{R} = \{z : \mathrm{Im}\, z < \frac{1}{2}\}$ and $\mathcal{R}' = \{z : |z-1| < 2\}$. Give an explicit formula for $f(z)$.*



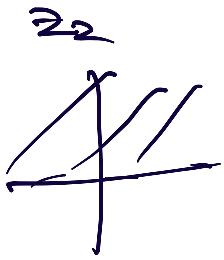


Solution:



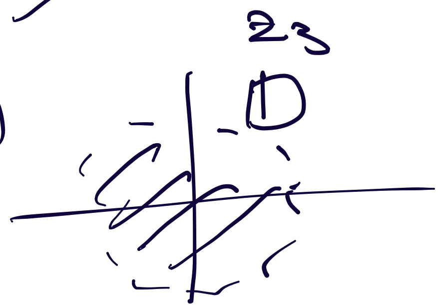
$\xrightarrow{z_1 = z - i/2}$

$\xrightarrow{z_2 = -z_1}$

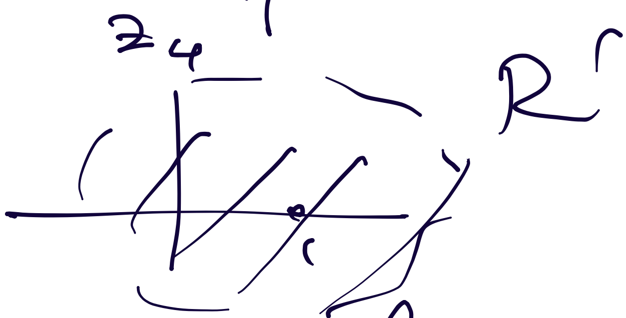


$\xrightarrow{z_3 = M_c(z_2)}$

M_c is the Cayley map



$\xrightarrow{z_4 = \cancel{2}\, 2z_3+1}$



all maps are biholo. and as such so is their composition. We can choose

$$f(z) = 2M_c\left(-\left(z - \tfrac{i}{2}\right)\right) + 1$$

$$\underset{\text{compare}}{=} \frac{6z - i}{2z - 3i}$$

Q.4 *(Assignment sheet 8 problem 3)*

(i) Show that for any $\rho > 0$ the sequence $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converges uniformly on $A = \{z \in \mathbb{C} : |z| \geq \rho\}$.

(ii) Does $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converge uniformly on $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$?

Solution: For any $z \in \mathbb{C}^*$

$$\lim_{n\to\infty} \frac{1}{nz} = \frac{1}{z} \lim_{n\to\infty} \frac{1}{n} = 0$$

i.e $f_n(z)$ converges pointwise to $f(z) = 0$.

(i) $$|f_n(z) - f(z)| = \frac{1}{n|z|} \underset{z\in A}{\leq} \frac{1}{n\rho} = s_n$$

$s_n \geq 0$ $\forall n$ and $s_n \xrightarrow[n\to\infty]{} 0$. s_n is indep. on z $\Rightarrow$ by a lemma from class $f_n(z)$ converges to $f(z)$ uniformly on A.

(ii) for $z_n = 1/n \in \mathbb{C}^*$ we have that

$$|f_n(z_n) - f(z_n)| = \frac{1}{n z_n} = 1 > 0$$

by a lemma from class $f_n(z)$ doesn't converge to $f(z)$ uniformly on $\mathbb{C}^*$.

Notice: (i) was true for any $\rho > 0$. This implies that f_n converges to f locally uniformly on $\mathbb{C}^*$.

$(\forall z_0 \in \mathbb{C}^*$ $\exists \rho_{z_0}$ s.t $z_0 \in \{z \mid |z| > \rho_{z_0}\}$

Q.5 *(Assignment sheet 8 problem 6) For every $n \in \mathbb{N}$, let $f_n(z) = \sin(z/n)$ for $z \in \mathbb{C}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{C}$. Let ρ be a positive real number. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $A = \{z : |z| \le \rho\}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathbb{C}$.*

$\frac{z}{n} \xrightarrow[n\to\infty]{} 0 \quad \forall z \in \mathbb{C}$

as sine is cont.

$f_n(z) = \sin(\frac{z}{n}) \xrightarrow[n\to\infty]{} \sin(0) = 0 = f(z)$

For uniform convergence we need to estimate $|f_n(z) - f(z)|$.

Recall

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$z = x + iy$

$|f_n(z) - f(z)| = |\sin(\frac{z}{n})| = \sqrt{\sin^2(\frac{x}{n})\cosh^2(\frac{y}{n}) + \cos^2(\frac{x}{n})\sinh^2(\frac{y}{n})}$

on $\{|z| \le \rho\}$ we get

$|x|, |y| \le \rho$

$\sin^2(\frac{x}{n}) \le |\frac{x}{n}|^2 \le (\rho/n)^2$

$\cosh^2(\frac{y}{n}) \le \cosh^2(\frac{|y|}{n}) \underset{\text{increasing}}{\le} \cosh^2(\rho/n)$

$\sinh^2(\frac{y}{n}) \underset{\text{similarly}}{\le} \sinh^2(\rho/n)$

that is enough.

(see written solution)

Q.6 *(Assignment sheet 8 problem 9) Prove that* $\sum_{n=0}^{\infty} e^{nz}$ *converges uniformly on* $A = \{z \in \mathbb{C} : \textit{Re}(z) \leq -1\}$*, but not on* $B = \{z \in \mathbb{C} : \textit{Re}(z) \leq 0\}$.

see written solution

Q.7 *(Assignment sheet 8 problem 10) Let R satisfy $0 < R < 1$. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$ converges uniformly on $A = \{z \in \mathbb{C} : |z| < R\}$. Conclude that the infinite series defines a continuous function on the unit disc $\mathbb{D}$.*

see written solution