Q.1 *(Assignment sheet 7 problem 4) Find a Möbius transformation f from the upper half-plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$ that takes $1+i$ to 0 and (when considered as a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) also takes 1 to $-i$. Give an explicit formula for $f(z)$.*

S.1 Denoting by M_C the Cayley map, we know that $(M_C)^{-1} = M_{C^{-1}}$ takes $\mathbb{D}$ to $\mathbb{H}$. Consequently, the Möbius map $M = M_{C^{-1}} \circ f$ takes $\mathbb{H}$ to itself. By the H2H theorem we know that we can find $S \in SL_2(\mathbb{R})$ such that

$$M_{C^{-1}} \circ f = M_S$$

and consequently (using the fact that $M_C \circ M_{C^{-1}} = M_{Id} = Id$) we have that

$$f = M_C \circ M_S = M_{CS}.$$

Writing $f = M_T$ for some $T \in GL_2(\mathbb{C})$ and using the fact that $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ we see that if $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ we can choose T to be

$$T = \begin{pmatrix} a - ic & b - id \\ a + ic & b + id \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. As f takes $1+i$ to 0 and 1 to $-i$ we have that

$$(a - ic)(1 + i) + b - id = 0$$

and

$$a - ic + b - id = -i(a + ic + b + id)$$

we find that

$$-i(a - ic) = -i(a + ic + b + id)$$

which implies that $-2ic = b + id$ and $a = \frac{d-b}{2}$. Lastly, we see that

$$1 = ad - bc = \frac{d^2 - bd}{2} + \frac{db - ib^2}{2} = \frac{d^2 - ib^2}{2}.$$

We conclude that $d = \pm\sqrt{2}$, $b = 0$ and consequently $a = \frac{\pm\sqrt{2}}{2}$ and $c = \mp\frac{\sqrt{2}}{2}$. Plugging this back yields

$$T = \pm\frac{\sqrt{2}}{2}\begin{pmatrix} 1 + i & -2i \\ 1 - i & 2i \end{pmatrix}$$

which, due to the scaling invariance, shows that

$$f(z) = \frac{(1 + i)z - 2i}{(1 - i)z + 2i}.$$

Q.2 *(Assignment sheet 7 problem 8) Use standard examples to find a biholomorphic map from the upper half $\Omega := \{z \in \mathbb{D} : Im(z) > 0\}$ of the unit disc onto the unit disc $\mathbb{D}$.*

S.2 We have seen that the Cayley map M_C takes the first quadrant $\Omega_1 = \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) > 0, \mathrm{Re}(z) > 0\}$ to the lower half of $\mathbb{D}$. We conclude that $g(z) = -M_C(z)$ takes Ω_1 to the upper half of $\mathbb{D}$ and consequently $g^{-1}(z) = M_{C^{-1}}(-z)$ takes the upper half of $\mathbb{D}$ to Ω_1. Continuing on the above, we notice that z^2 is a biholomrphic map that takes Ω_1 to $\mathbb{H}$ and by using the Cayley map again we see that

$$f(z) = M_C\left(\left(g^{-1}(-z)\right)^2\right) = M_C\left(M_{C^{-1}}(-z)^2\right)$$

takes the upper half of $\mathbb{D}$ to $\mathbb{D}$ in a bioholomorphic way. Using the fact that $M_C(z) = \frac{z-i}{z+i}$ and $M_{C^{-1}}(z) = \frac{iz+i}{-z+1}$ we find that

$$f(z) = \frac{(z - 1)^2 + i(z + 1)^2}{(z - 1)^2 - i(z + 1)^2}.$$

Q.3 *(Assignment sheet 7 problem 11) Construct a biholomorphic map f from $\mathcal{R}$ onto $\mathcal{R}'$, where $\mathcal{R} = \{z : Im\, z < \frac{1}{2}\}$ and $\mathcal{R}' = \{z \;:\; |z-1| < 2\}$. Give an explicit formula for $f(z)$.*

S.3 We see that we are asked to map half a plane to a circle so the Cayley map pops to mind. Consider the map $f_1(z) = z - \frac{i}{2}$. We have that f_1 takes $\mathcal{R}$ to the lower half plane and as such $-f_1$ takes $\mathcal{R}$ to $\mathbb{H}$. We conclude that

$$f_2(z) = M_C\left(-f_1(z)\right)$$

takes $\mathcal{R}$ to $\mathbb{D}$. Next we notice that $f_3(z) = \frac{z-1}{2}$ takes $\mathcal{R}'$ to $\mathbb{D}$ as well and is invertible with $f_3^{-1}(z) = 2z+1$. As all the above maps are biholomorphic we find that

$$f(z) = 2M_C\left(-z + \frac{i}{2}\right) + 1$$

takes $\mathcal{R}$ to $\mathcal{R}'$ in a biholomorphic way. Using the fact that $M_C(z) = \frac{z-i}{z+i}$ we find that

$$f(z) = \frac{6z-i}{2z-3i}.$$

Q.4 *(Assignment sheet 8 problem 3)*

(i) Show that for any $\rho > 0$ the sequence $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converges uniformly on $A = \{z \in \mathbb{C} :\; |z| \geq \rho\}$.

(ii) Does $\left\{\frac{1}{nz}\right\}_{n\in\mathbb{N}}$ converge uniformly on $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$?

S.4 (i) For all $z \in \mathbb{C}^*$ we have that

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{nz} = 0$$

and as such $\{f_n(z)\}_{n\in\mathbb{N}}$ converges pointwise to $f(z) = 0$. On A we find that

$$|f_n(z) - f(z)| = \frac{1}{n|z|} \leq \frac{1}{n\rho} = s_n.$$

We know that $\{s_n\}_{n\in\mathbb{N}}$ is a positive sequence that is independent of z and converges to zero. We conclude, according to a lemma from class, that $\{f_n(z)\}_{n\in\mathbb{N}}$ converges to zero uniformly on A.
(ii) The convergence on $\mathbb{C}^*$ will not be uniform as we can get as close as we want to $z = 0$ where the sequence of functions is unbounded. Indeed, choosing $z_n = \frac{1}{n}$ we see that $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}^*$ and

$$|f_n(z_n) - f(z_n)| = 1.$$

We conclude, according to a lemma from class, that $\{f_n(z)\}_{n\in\mathbb{N}}$ does not converge to zero uniformly on A.

Q.5 *(Assignment sheet 8 problem 6) For every $n \in \mathbb{N}$, let $f_n(z) = \sin(z/n)$ for $z \in \mathbb{C}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{C}$. Let ρ be a positive real number. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $A = \{z : |z| \leq \rho\}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathbb{C}$.*

S.5 We know that for any (fixed) $z \in \mathbb{C}$

$$\lim_{n\to\infty} \frac{z}{n} = 0$$

(not uniformly!). As $\sin(z)$ is a continuous function we conclude that for any (fixed) $z \in \mathbb{C}$

$$\lim_{n\to\infty} \sin\left(\frac{z}{n}\right) = \sin(0) = 0,$$

showing that $f_n(z)$ converges to $f(z) = 0$ pointwise on $\mathbb{C}$.
To show the uniform convergence we recall that

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

and as such
$$f_n(z) = \sin\left(\frac{z}{n}\right) = \sin\left(\frac{x}{n}\right)\cosh\left(\frac{y}{n}\right) + i\cos\left(\frac{x}{n}\right)\sinh\left(\frac{y}{n}\right).$$
We conclude that
$$|f_n(z) - f(z)| = \sqrt{\sin^2\left(\frac{x}{n}\right)\cosh^2\left(\frac{y}{n}\right) + \cos^2\left(\frac{x}{n}\right)\sinh^2\left(\frac{y}{n}\right)}.$$
At this point it is important to (yet again) remind ourselves that $\sin(z)$ is not bounded in general due to the appearance of $\sinh$ and $\cosh$. We do find that
$$|f_n(z) - f(z)| \leq \sqrt{\sin^2\left(\frac{x}{n}\right)\cosh^2\left(\frac{y}{n}\right) + \sinh^2\left(\frac{y}{n}\right)}$$
(where we have kept the terms that we know will go to zero) and using the facts that $\sin(-x) = -\sin(x)$, $\sinh(-x) = \sinh(x)$, and $\cosh(-x) = \cosh(x)$ we find that
$$|f_n(z) - f(z)| \leq \sqrt{\sin^2\left(\left|\frac{x}{n}\right|\right)\cosh^2\left(\left|\frac{y}{n}\right|\right) + \sinh^2\left(\left|\frac{y}{n}\right|\right)}$$
Using the fact that on A
$$\max\{|x|, |y|\} \leq |z| \leq \rho$$
together with the facts that $\sinh$ and $\cosh$ are increasing on $[0,\infty)$ and $|sin(x)| \leq |x|$ for all $x \in \mathbb{R}$ we find that on A
$$|f_n(z) - f(z)| \leq \sqrt{\left(\frac{\rho}{n}\right)^2\cosh^2\left(\frac{\rho}{n}\right) + \sinh^2\left(\frac{\rho}{n}\right)} = s_n.$$
We know that $\{s_n\}_{n\in\mathbb{N}}$ is a positive sequence that is independent of z and converges to zero. We conclude, according to a lemma from class, that $\{f_n(z)\}_{n\in\mathbb{N}}$ converges to zero uniformly on A.
The lack of uniform convergence on $\mathbb{C}$ can be seen by considering the sequence $z_n = n$. Indeed
$$|f_n(z_n) - f(z_n)| = |\sin(1)| \underset{n\to\infty}{\not\longrightarrow} 0.$$
We can do "worse" and even get unboundedness. For instance, choosing $z_n = in^2$ (to bring out the $\sinh$ and $\cosh$ which only depend on $\mathrm{Im}(z)$) gives us
$$|f_n(z_n) - f(z_n)| = |\sinh(n)| = \sinh(n)$$
which goes to infinity as n goes to infinity.

Q.6 *(Assignment sheet 8 problem 9) Prove that $\sum_{n=0}^{\infty} e^{nz}$ converges uniformly on $A = \{z \in \mathbb{C} : Re(z) \leq -1\}$, but not on $B = \{z \in \mathbb{C} : Re(z) \leq 0\}$.*

S.6 For any $z \in A$ we have that
$$|e^{nz}| = e^{n\,\mathrm{Re}(z)} \leq e^{-n} = \left(e^{-1}\right)^n = M_n.$$
As $\sum_{n=0}^{\infty} M_n < \infty$ (a geometric series with $q = e^{-1} < 1$) we conclude by Weierstrass' M-test that the series converges uniformly on A.
The reason we will have issues with the convergence on B is the fact that the real part of z can be zero. Indeed, $z = 0$ is in B and
$$\sum_{n=0}^{\infty} e^{n\cdot 0} = \sum_{n=0}^{\infty} 1 = \infty$$
so the series doesn't even converge on B.
It is worth to mention that the series does *converge locally uniformly on* $B^0 = \{z \in \mathbb{C} : Re(z) < 0\}$. Indeed, given any $z \in B^0$ we consider the open set $U_z = \left\{w \in \mathbb{C} \ : \ \mathrm{Re}(w) < \frac{\mathrm{Re}(z)}{2}\right\}$. Clearly $z \in U_z$ and for any $w \in U_z$ we have that
$$|e^{nw}| = e^{n\,\mathrm{Re}(w)} \leq e^{\frac{n\,\mathrm{Re}(z)}{2}} = \left(e^{-\frac{|\mathrm{Re}(z)|}{2}}\right)^n = M_n(U_z).$$
As $\sum_{n=0}^{\infty} M_n(U_z) < \infty$ we conclude the result by using Weierstrass' local M-test.

Q.7 *(Assignment sheet 8 problem 10) Let* R *satisfy* $0 < R < 1$. *Show that the series* $\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$ *converges uniformly on* $A = \{z \in \mathbb{C} : |z| < R\}$. *Conclude that the infinite series defines a continuous function on the unit disc* $\mathbb{D}$.

S.7 We recall that the reverse triangle inequality states that

$$|z - w| \geq ||z| - |w||.$$

Using the above inequality, we conclude that if $|z| < R < 1$ then

$$|1 + z^n| = |1 - (-z^n)| \geq |1 - |-z^n|| = |1 - |z|^n| = 1 - |z|^n \geq 1 - R^n,$$

where we used the fact that $|z|^n < R^n < 1$ as $R < 1$.
For any $z \in A$ we have that

$$\left|\frac{z^n}{1+z^n}\right| = \frac{|z^n|}{|1+z^n|} \leq \frac{R^n}{1-R^n} = M_n.$$

Since $M_n > 0$ and

$$\frac{M_{n+1}}{M_n} = \frac{1-R^n}{1-R^{n+1}}\, R \underset{n\to\infty}{\longrightarrow} \frac{1-0}{1-0}\, R = R < 1$$

we conclude that $\sum_{n=0}^{\infty} M_n < \infty$. Using Weierstrass' M-test, we find that $\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$ converges uniformly on A. Since $\frac{z^n}{1+z^n}$ are continuous on A (as we do not include the boundary $|z| = 1$ where the denominator is unbounded) we conclude that the limit function of the series is continuous by the same theorem.