Q.1 *Show that for any $a \in \mathbb{C}$ the curve $\gamma(t) = e^{\alpha t}$ satisfies*

$$\gamma'(t) = \alpha e^{\alpha t} (= \alpha \gamma(t)).$$

Conclude that for any $-\infty < a < b < \infty$ and any $\alpha \neq 0$

$$\int_a^b e^{\alpha t} dt = \frac{e^{\alpha b} - e^{\alpha a}}{\alpha}.$$

<u>Sketch:</u> By def

$$\gamma'(t) = (\mathrm{Re}\,\gamma(t))' + i(\mathrm{Im}\,\gamma(t))'$$

if $\alpha = \alpha_1 + i\alpha_2$

$$\gamma(t) = e^{(\alpha_1 + i\alpha_2)t} = e^{\alpha_1 t} \cdot e^{i\alpha_2 t}$$

$$= \underbrace{e^{\alpha_1 t} \cos(\alpha_2 t)}_{\mathrm{Re}\,\gamma(t)} + i \underbrace{e^{\alpha_1 t} \sin(\alpha_2 t)}_{\mathrm{Im}\,\gamma(t)}$$

Following the def gives the result

$$(\mathrm{Re}\,\gamma(t))' = \alpha_1 e^{\alpha_1 t} \cos(\alpha_2 t) - \alpha_2 e^{\alpha_1 t} \sin(\alpha_2 t)$$

$$(\mathrm{Im}\,\gamma(t))' = \alpha_1 e^{\alpha_1 t} \sin(\alpha_2 t) + \alpha_2 e^{\alpha_1 t} \cos(\alpha_2 t)$$

$$(\mathrm{Re}\,\gamma(t))' + i(\mathrm{Im}\,\gamma(t))' = e^{\alpha_1 t}\Big[(\alpha_1 + i\alpha_2)\cos(\alpha_2 t)$$

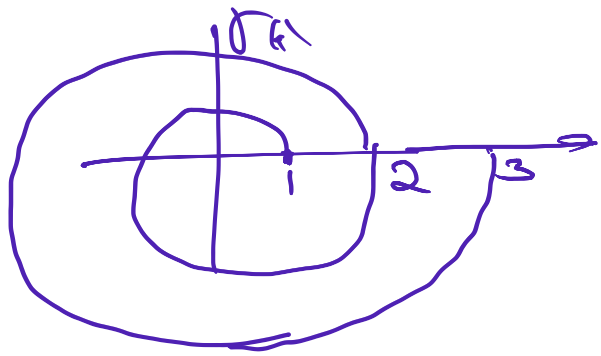
$$+ \underbrace{(-\alpha_2 + i\alpha_1)}_{i(\alpha_1 + i\alpha_2)} \sin(\alpha_2 t)\Big] = (\alpha_1 + i\alpha_2) e^{\alpha_1 t} \cdot \big(\cos(\alpha_2 t) + i \sin(\alpha_2 t)\big)$$

$$= \alpha e^{\alpha t}$$

The second part follows (like in FTC-I) from the fact that for $\alpha \neq 0$

$$\int_a^b e^{\alpha t} dt = \frac{1}{\alpha} \int_a^b \frac{d}{dt}\left(e^{\alpha t}\right) dt = \ldots$$

Q.2 *(Assignment sheet 9 problem 5) Calculate* $\int_\gamma \frac{1}{z}\,dz$*, where* $\gamma(t) = (1+2t)e^{4\pi i t}$ *for* $0 \le t \le 1$.



$1/z$ has a holo. anti-derivatives in domains that have a branch cut for $\log z$

$\gamma(t)$ doesn't belong to any such domain.

We need to calculate by def.

$$\gamma'(t) = 2\cdot e^{4\pi i t} + (1+2t)\cdot 4\pi i e^{4\pi i t}$$

product rule

$$= \left[2 + (1+2t)\cdot 4\pi i\right] e^{4\pi i t}$$

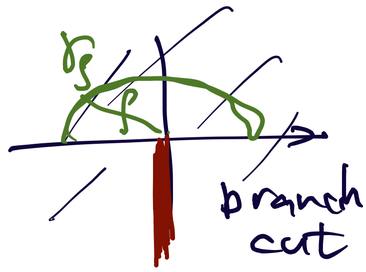
$$\int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{1}{\gamma(t)}\gamma'(t)\,dt = \int_0^1 \frac{(2+(1+2t)4\pi i)\cancel{e^{4\pi i t}}}{(1+2t)\cancel{e^{4\pi i t}}}\,dt$$

$$= \int_0^1 \left[\frac{2}{1+2t} + 4\pi i\right]dt = \underset{\text{real}}{\log}(1+2t)\Big|_0^1 + 4\pi i t\Big|_0^1$$

$$= \log 3 + 4\pi i$$

Q.3 *(Assignment sheet 9 problem 6) Let γ_ρ be the curve $\gamma_\rho(\theta) := \rho e^{i\theta}$ with $0 \le \theta \le \pi$. Let $z^{\frac{1}{2}}$ be the branch of square root corresponding to the branch of* log *with argument in $(-\pi/2, 3\pi/2)$, that is, if $z = \rho e^{i\theta}$ with $\theta \in (-\pi/2, 3\pi/2)$ then $z^{\frac{1}{2}} = \sqrt{\rho}e^{i\theta/2}$. Show that*

$$\lim_{\rho\to\infty}\int_{\gamma_\rho}\frac{z^{1/2}}{z^2+1}dz = 0.$$



$z^{1/2}$ is holo in $\mathbb{C}\setminus i\mathbb{R}_{\le 0}$.

γ_ρ in the domain of holo.

No idea how to find anti-derivative for $\frac{z^{1/2}}{z^2+1}$

We will use the estimation lemma.

$$\left|\int_{\gamma_\rho}\frac{z^{1/2}}{z^2+1}dz\right| \le \sup_{\gamma_\rho}\frac{|z^{1/2}|}{|z^2+1|}L(\gamma_\rho)$$

$$\gamma_\rho'(\theta) = i\rho e^{i\theta}$$

$$|\gamma_\rho'(\theta)| = \rho \to L(\gamma_\rho) = \int_0^\pi \rho\, d\theta = \pi\rho$$

$$|z^{1/2}| \underset{\substack{z = re^{i\theta} \\ z^{1/2} = \sqrt{r}e^{i\theta/2}}}{=} |z|^{1/2} \underset{\text{on } \gamma_\rho}{=} \sqrt{\rho}$$

$$|z^2+1| = |z^2-(-1)| \ge \big||z^2| - |-1|\big| = \big||z|^2-1\big|$$

$$|z-w| \ge \big||z|-|w|\big|$$

$$\underset{z\in\gamma_\rho}{=} |\rho^2-1| \underset{\substack{\text{so } \rho\to\infty \\ \rho>1}}{=} \rho^2-1$$

$$\Rightarrow \forall z\in\gamma_\rho \quad \frac{|z^{1/2}|}{|z^2+1|} \le \frac{\sqrt{\rho}}{\rho^2-1}$$

We conclude that

$$0 \le \left|\int_{\gamma_\rho}\frac{z^{1/2}}{z^2+1}dz\right| \le \sup_{z\in\gamma_\rho}\frac{|z^{1/2}|}{|z^2+1|}L(\gamma_\rho) \le \frac{\sqrt{\rho}}{\rho^2-1}\cdot\pi\rho$$

$\rho\to\infty$ by pinching

0 $\qquad\qquad\qquad\qquad \downarrow \rho\to\infty$

0

Q.4 *(Assignment sheet 9 problem 7) Let γ be any piecewise C^1-curve from -3 to 3 such that, except for the end points, lies entirely in the upper half plane. Calculate the integral*

$$\int_\gamma f(z)\,dz,$$

where $f(z)$ is the branch of $z^{\frac{1}{2}}$ defined by $\sqrt{r}e^{i\theta/2}$ with $0<\theta<2\pi$.

There is an issue here since the branch cut of $z^{1/2}$ is where the curve ends. We can't find a holo. anti-derivative that will be defined on $\mathbb{R}_{\geq 0}$ and as such we can't use FTC.

This only happens at one point of the curve. if $\gamma:[a,b]\to\mathbb{C}$ then since $z^{1/2}$ is bounded near $z=3$

$$\int_{[a,b]} f(\gamma(t))\gamma'(t)\,dt = \int_{[a,b)} f(\gamma(t))\gamma'(t)\,dt$$

We define $f_1(z)=z^{1/2}=\sqrt{r}e^{i\theta/2}$ when $z=re^{i\theta}$ and $\theta\in(-\pi/2, 3\pi/2)$.

on $\mathbb{H}$ $f_1(z)=f_2(z)$ ($\theta\in[0,\pi]$ in both cases)

$$\Rightarrow \int_\gamma f(z)\,dz = \int_\gamma f_1(z)\,dz$$

$\gamma\in\mathbb{H}$ besides the end point

$f_1(z)$ has an anti-derivative in $\mathbb{C}\setminus i\mathbb{R}_{\leq 0}$ which includes γ.

$F_1(z) = \frac{2}{3} z^{3/2} = \frac{2}{3} r^{3/2} e^{i\frac{3\theta}{2}} \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

is the anti-derivative for f_1

$\Rightarrow \int_\gamma f(z)\,dz = \int_\gamma f_1(z)\,dz = F_1(\gamma(b)) - F_1(\gamma(a))$

$= F_1(3) - F_1(-3)$

$= \ldots$