Q.1 *Show that for any $a \in \mathbb{C}$ the curve $\gamma(t) = e^{\alpha t}$ satisfies*

$$\gamma'(t) = \alpha e^{\alpha t} (= \alpha\gamma(t)).$$

Conclude that for any $-\infty < a < b < \infty$ and any $\alpha \neq 0$

$$\int_a^b e^{\alpha t} dt = \frac{e^{\alpha b} - e^{\alpha a}}{\alpha}.$$

S.1 Writing

$$\alpha = \alpha_1 + i\alpha_2$$

we find that

$$e^{\alpha t} = e^{\alpha_1 t} \cos(\alpha_2 t) + i e^{\alpha_1 t} \cos(\alpha_2 t)$$

which means that

$$\frac{d}{dt} e^{\alpha t} = \frac{d}{dt}\left(e^{\alpha_1 t}\cos(\alpha_2 t)\right) + i\frac{d}{dt}\left(e^{\alpha_1 t}\cos(\alpha_2 t)\right)$$
$$= \alpha_1 e^{\alpha_1 t} cos(\alpha_2 t) - \alpha_2 e^{\alpha_1 t}\sin(\alpha_2 t) + i\left(\alpha_1 e^{\alpha_1 t} sin(\alpha_2 t) + \alpha_2 e^{\alpha_1 t}\cos(\alpha_2 t)\right)$$
$$= (\alpha_1 + i\alpha_2)\, e^{\alpha_1 t}\cos(\alpha_2 t) + (-\alpha_2 + i\alpha_1)\, e^{\alpha_1 t}\sin(\alpha_2 t)$$
$$= (\alpha_1 + i\alpha_2)\, e^{\alpha_1 t}(\cos(\alpha_2 t) + i\sin(\alpha_2 t)) = \alpha e^{\alpha t}.$$

To second statement follows from the fact that

$$\int_a^b e^{\alpha t} dt = \frac{\int_a^b \frac{d}{dt}\left(e^{\alpha t}\right) dt}{\alpha} = \frac{e^{\alpha b} - e^{\alpha a}}{\alpha}.$$

Q.2 *(Assignment sheet 9 problem 5) Calculate $\int_\gamma \frac{1}{z}\, dz$, where $\gamma(t) = (1+2t)e^{4\pi i t}$ for $0 \leq t \leq 1$.*

S.2 Using the product rule we find that

$$\gamma'(t) = 2e^{4\pi i t} + 4\pi i\,(1+2t)\, e^{4\pi i t} = (2 + 4\pi i\,(1+2t))\, e^{4\pi i t}$$

and consequently

$$\int_\gamma \frac{1}{z}\, dz = \int_0^1 \frac{(2 + 4\pi i\,(1+2t))\, e^{4\pi i t}}{(1+2t)e^{4\pi i t}} dt = \int_0^1 \left(\frac{2}{1+2t} + 4\pi i\right) dt$$
$$= (\log(1+2t) + 4\pi i t)\,|_0^1 = \log 3 + 4\pi i.$$

Q.3 *(Assignment sheet 9 problem 6) Let γ_ρ be the curve $\gamma_\rho(\theta) := \rho e^{i\theta}$ with $0 \leq \theta \leq \pi$. Let $z^{\frac{1}{2}}$ be the branch of square root corresponding to the branch of* $\log$ *with argument in $(-\pi/2, 3\pi/2)$, that is, if $z = \rho e^{i\theta}$ with $\theta \in (-\pi/2, 3\pi/2)$ then $z^{\frac{1}{2}} = \sqrt{\rho} e^{i\theta/2}$. Show that*

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} \frac{z^{1/2}}{z^2+1} dz = 0.$$

S.3 Using the estimate lemma we know that

$$\left|\int_{\gamma_\rho} \frac{z^{1/2}}{z^2+1} dz\right| \leq \sup_{\gamma_\rho} \left|\frac{z^{\frac{1}{2}}}{z^2+1}\right| L(\gamma_\rho).$$

We have that when $|z| = \rho$

$$\left|z^2+1\right| = \left|z^2 - (-1)\right| \geq \left|\left|z^2\right| - 1\right| = \left|\rho^2 - 1\right|$$

and using the fact that $L(\gamma_\rho) = \pi\rho$ we find that for any $\rho \neq 1$

$$\left|\int_{\gamma_\rho} \frac{z^{1/2}}{z^2+1} dz\right| \leq \pi\rho \frac{\sqrt{\rho}}{|\rho^2 - 1|}.$$

Since

$$\lim_{\rho\to\infty} \pi\rho \frac{\sqrt{\rho}}{|\rho^2 - 1|} = \pi\rho \frac{\sqrt{\rho}}{\rho^2 - 1} = 0$$

we conclude by the pinching lemma that

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} \frac{z^{1/2}}{z^2+1} dz = 0.$$

Q.4 *(Assignment sheet 9 problem 7) Let γ be any piecewise C^1-curve from -3 to 3 such that, except for the end points, lies entirely in the upper half plane. Calculate the integral*

$$\int_\gamma f(z)\, dz,$$

where $f(z)$ is the branch of $z^{\frac{1}{2}}$ defined by $\sqrt{r}e^{i\theta/2}$ with $0 < \theta < 2\pi$.

S.4 We first notice that the problem is a bit more delicate than it seems as one of the end point of such curves lies on the branch cut of our function. Formally speaking this can be circumvented by noticing that $f(z)$ remains bounded near $z = 3$ and consequently, given $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(b) = 3$ we will have that

$$\int_{[a,b)} f(\gamma(t))\, \gamma'(t) dt = \int_{[a,b]} f(\gamma(t))\, \gamma'(t) dt$$

since $f \circ \gamma$ is continuous on $[a, b)$ and remains bounded as t approaches b.
Since γ is not known we suspect that we will need to use the FCT. To avoid having to deal with the given cut we notice that if we consider f_1 to be the branch cut of $z^{\frac{1}{2}}$ defined by $f_1(z) = \sqrt{r}e^{i\theta/2}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ then

$$f_1 \circ \gamma = f \circ \gamma$$

on $[a, b)$ and $f_1(z)$ has an anti-derivative in $\mathbb{C} \setminus i\mathbb{R}_{\leq 0}$ given by

$$F_1(z) = \frac{2}{3} z^{\frac{3}{2}} = \frac{2}{3}\sqrt{r}^3 e^{i\frac{3\theta}{2}}$$

with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Using the FTC we conclude that

$$\int_\gamma f(z)dz = \int_{[a,b)} f(\gamma(t))\, \gamma'(t)dt = \int_a^b f_1(\gamma(t))\, \gamma'(t)dt = F_1(\gamma(b)) - F_1(\gamma(a))$$

$$= F_1(3) - F_1(-3) = \frac{2}{3}\left(\sqrt{27}e^{i\frac{3\cdot 0}{2}} - \sqrt{27}e^{i\frac{3\cdot\pi}{2}}\right) = 2\sqrt{3}\,(1+i)\,.$$