The following questions are all taken from various past exam papers for Complex Analysis II. These questions have been selected to complement the May/June 2024 exam.

Q.1 *[Q3 2016]*

(a) Define a metric space X. What is an open, respectively closed, set in X?

(b) Let x and y be two different points in a metric space X. Show that there exist two open disjoint sets containing x and y respectively.

Q.2 *[Q1 2017]*

(a) Suppose (X, d) is a metric space. What does it mean for (X, d) to be sequentially compact*?*

(b) Show that the open unit interval $(0, 1)$ with its usual metric is not compact.

Q.3 *[Q8 2019]*

(a) [Note, this question part is a corrected version of the one visible in the exam]
Find a transformation taking the region $\mathcal{R}_1 = \{z \,:\, |z| < 1, \mathrm{Im(z)} < 0\}$ (the lower half of the unit disc) to the upper half plane $\mathbb{H} = \{z \,:\, \mathrm{Im}(z) > 0\}$.

(b) Find a conformal map that maps the region $\mathcal{R}_1$ to $\mathcal{R}_2 = \{z \,:\, |z| < 1\} \setminus \mathbb{R}_{\leq 0}$ (the unit disc with the non positive reals removed).

(c) Find the image of $\mathcal{R}_2$ under the principal branch of log.

Q.4 *[Q9, 2004]*

(a) Determine all Möbius transformations T for which $T(\infty) = \infty$ and $T(1) - T(0) = 1$. What is the geometric meaning of T and of its inverse T^{-1}?

(b) Let C_1 be the circle passing through $0, 1, -i$ and C_2 be the circle passing through $0, 1, i$. Let Ω be the intersection of the two discs bounded by C_1 and C_2.

(i) Determine the unique Möbius transformation S which maps the ordered set of points $\{0, 1, -i\}$ to the ordered set of points $\{-1, \infty, i\}$.

(ii) Sketch the image of Ω under S.

Q.5 *[Q5 2022]*

(a) Let $U = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ and $V = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that exp is a conformal in $\mathbb{C}$ and satisfies $f(U) = V$. Is exp: $U \to V$ a biholomorphism?

(b) Using part (a) or otherwise, find a conformal map from $\{z \in \mathbb{C} : |z| < 1\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$.

Q.6 *[Q2.3 2021]*

Suppose that $t > 0$. Prove that $f_n : \{z \in \mathbb{C} :\, \mathrm{Re}(z) \geq t\} \to \mathbb{C}$ defined by

$$f_n(z) := \tanh(nz) = \frac{\sinh(nz)}{\cosh(nz)}$$

is uniformly convergent to 1 as $n \to \infty$. Is the convergence uniform in $\{z \in \mathbb{C} :\, \mathrm{Re}(z) > 0\}$? Justify your answer.

Q.7 *[Q3.2 2020 Resit]*

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{3^n + z^n}$$

is uniformly convergent on $|z| \leq \rho$ for every real ρ with $0 < \rho < 3$. Is the convergence uniform on $|z| < 3$? [Hint (which was not provided in the original exam): You may use without proof the fact that if $f_n(z)$ doesn't converge uniformly to zero on a set U, then the series $\sum_{n=1}^{\infty} f_n(z)$ can't converge uniformly]

Q.8 *[Q3.3 2021]*

Let p be a polynomial with complex coefficients. Using a parametrisation of the unit circle, show that

$$\overline{p'(0)} = \frac{1}{2\pi i}\int_{|z|=1} \overline{p(z)}\, dz.$$

Find $\int_{|z|=1} \mathrm{Re}(p(z))\, dz$.

Q.9 (a) *[Q9a 2012]*

Show that there exists an unbounded open subset $S \subset \mathbb{C}$ on which $\sin(z)$ is bounded.

(b) *[Q6 2011]*

Show that there exists no holomorphic function f such that $f(z) = |\sin(z)|$ for all purely real $z = x$ with $-1 < x < 1$.

Q.10 *[Q1.4 2020]*

Let f be a holomorphic function on $\mathbb{C} - \{0\}$. Show that f is bounded if and only if f is constant. State clearly any results you use from lectures.

Q.11 *[Q3.1 2022 Resit]*

Consider the meromorphic function $f(z) = \dfrac{1}{z^2(8+z^3)}$.

Determine the Laurent series expansion of $f(z)$ on the annulus $\mathcal{A} = \{z \in \mathbb{C} : 0 < |z| < 2\}$.

Q.12 *[Q4 2019]*

(i) *Find all the zeros and poles, with their orders, of* $f(z) = \dfrac{z}{\sin z + \cos z}$.

(ii) *Find the residue of f at each of its poles.*

Q.13 *[Q9 2016]*

Consider the function $g(z) = \dfrac{e^{-z^2}}{1+e^{-2az}}$, *where* $a = (1+i)\frac{\sqrt{\pi}}{\sqrt{2}} = e^{i\pi/4}\sqrt{\pi}$ *is fixed.*

(a) *Show $a^2 = i\pi$ and $e^{-2a(z+a)} = e^{-2az}$. Use this to show that*

$$g(z) - g(z+a) = e^{-z^2}. \qquad (*)$$

(b) *Show that all poles of g occur at $z = \frac{a}{2} + na$ with $n \in \mathbb{Z}$. Compute the residue at $z = \frac{a}{2}$.*

(c) *For r and s positive real numbers, consider the contour γ given by the boundary of the parallelogram with vertices $s, s+a, -r+a$ and $-r$. Draw the contour marking all the poles of $g(z)$.*

(d) *Use $(*)$ to show that the horizontal line integrals of $\int_\gamma g(z)dz$ combine to $\int_{-r}^{s} e^{-x^2} dx$. Use this and Cauchy's residue theorem to find an expression for $\int_{-r}^{s} e^{-x^2} dx$.*

(e) *Conclude*

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Q.14 (a) *[Q1.4 2020]*

Show that the polynomial $z^5 + 15z + 1$ has precisely four zeros (counted with multiplicity) in the set $\{\, z : \frac{3}{2} \le |z| < 2\}$.

(b) *[Q5b 2018]*

Fix $R > 0$. Prove that if N is sufficiently large, depending on R, then $\displaystyle\sum_{k=0}^{N} \frac{z^k}{k!} = 0$ *has no solutions $z \in D(0,R)$. You can use any properties of the exponential function that you like, provided they are stated clearly.*