The following questions are all taken from various past exam papers for Complex Analysis II. These questions have been selected to complement the May/June 2024 exam.

Q.1 *[Q3 2016]*

(a) Define a metric space X. What is an open, respectively closed, set in X?

(b) Let x and y be two different points in a metric space X. Show that there exist two open disjoint sets containing x and y respectively.

S.1 (a) A metric space X is a set together with a "distance" function $d : X \times X \to \mathbb{R}$ such that

$$\begin{aligned} &d(x, y) \geq 0 \quad \text{with equality if and only if } x = y \\ &d(x, y) = d(y, x) \\ &d(x, y) \leq d(x, z) + d(z, y). \end{aligned}$$

A subset $U \subseteq X$ is open in X if for every $z \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(z) = \{y \in X : d(z, y) < \epsilon\} \subset U$. A subset $U \subseteq X$ is closed if its complement $X \setminus U$ is open in X.

(b) Let $r = d(x, y)$ be the distance between x and y. Then, since we know from lectures that open balls are open, $U_1 = B_{r/2}(x)$ and $U_2 = B_{r/2}(y)$ are two disjoint open sets which contain x and y respectively. Indeed, for $z \in U_1 \cap U_2$ we would have by the triangle inequality that

$$r = d(x, y) \leq d(x, z) + d(x, y) < r/2 + r/2 = r$$

which is a contradiction.

Q.2 *[Q1 2017]*

(a) Suppose (X, d) is a metric space. What does it mean for (X, d) to be sequentially compact*?*

(b) Show that the open unit interval $(0, 1)$ with its usual metric is not compact.

S.2 (a) X is sequentially compact if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X, there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ whose limit is in X.

(b) We give a sequence of points in $(0, 1)$ that converge to $0 \notin (0, 1)$. Consider the sequence of points $x_n = 1/n$ for $n \geq 2$. Since $d(0, x_n) = 1/n$ we have that $x_n \to 0$ as $n \to \infty$. So we must also have $x_{n_k} \to 0$ as $k \to \infty$ for any subsequence. But $0 \notin (0, 1)$. Thus there exists no subsequence of $\{x_n\}_{n\in\mathbb{N}}$ which has a limit in $(0, 1)$.

Q.3 *[Q8 2019]*

(a) [Note, this question part is a corrected version of the one visible in the exam]
Find a transformation taking the region $\mathcal{R}_1 = \{z : |z| < 1, \mathrm{Im(z)} < 0\}$ (the lower half of the unit disc) to the upper half plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$.

(b) Find a conformal map that maps the region $\mathcal{R}_1$ to $\mathcal{R}_2 = \{z : |z| < 1\} \setminus \mathbb{R}_{\leq 0}$ (the unit disc with the non positive reals removed).

(c) Find the image of $\mathcal{R}_2$ under the principal branch of log.

S.3 (a) We know the Cayley transform M_C is a map from $\mathbb{H}$ to $\mathbb{D}$, so its inverse $M_{C^{-1}}$ maps $\mathbb{D}$ to $\mathbb{H}$. To find the image of $\mathcal{R}_1$ under $M_{C^{-1}}$, we first consider how it acts on two segments of the boundary:

* The line segment from -1 to 1 (through 0). We have $M_{C^{-1}} = \frac{iz+i}{-z+1}$ so

$$M_{C^{-1}}(-1) = \frac{-i + i}{1 + 1} = 0, \quad M_{C^{-1}}(1) = \frac{i + i}{-1 + 1} = \infty, \quad M_{C^{-1}}(0) = \frac{0 + i}{0 + 1} = i.$$

Thus, the line segment from -1 to 1 which passes through 0 is taken to the line segment from 0 to ∞ which passes through i. Consequently, the image of the line segment is the non-negative imaginary axis.

* The circular arc from -1 to 1 which passes through $-i$. We have

$$M_{C^{-1}}(-i) = \frac{-i^2 + i}{i+1} = 1.$$

Thus, the circular arc from -1 to 1 which passes through $-i$ is taken to the line segment from 0 to ∞ which passes through 1. Consequently the image of the circular arc is the positive real axis. *[Instead, we could just have used conformality to deduce that this was the image - the angle and its orientation at $z = -1$ must be preserved, so the positive real axis had to be the image.]*

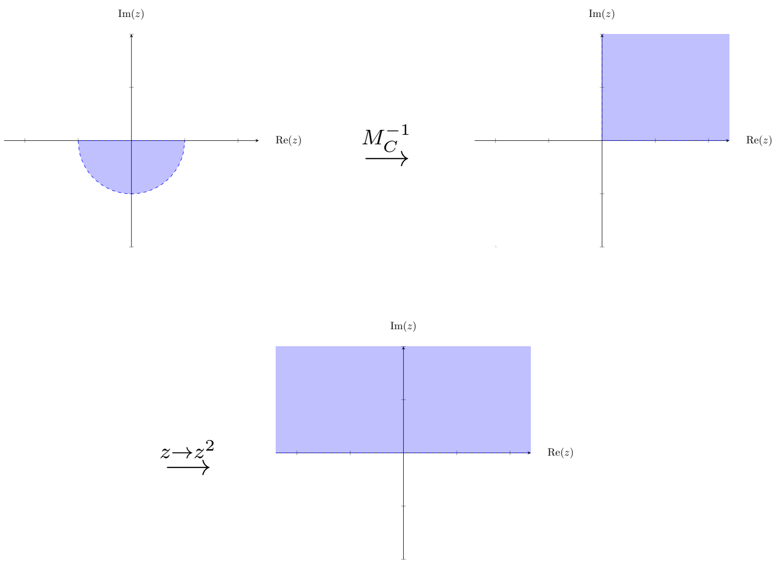
To see where the interior is mapped to, we choose a point in $\mathcal{R}_1$ and see where its image lies. For $\frac{-i}{2}$, we have that

$$M_{C^{-1}}(-i/2) = \frac{4+3i}{5}.$$

So combining our observations above, we conclude that the image of $\mathcal{R}_1$ under $M_{C^{-1}}$ is the first quadrant $\Omega = \{w \in \mathbb{C} : 0 < \mathrm{Arg}(w) < \pi/2\}$. [Alternatively, we could have concluded this by conformality as the interior must stay on the 'same side' of each line segment otherwise the orientation would be reversed.]

we notice that the map $g : \Omega \to \mathbb{H}$, $g : z \mapsto z^2$ open up the first quadrant to the upper half plane and conclude that the desired transformation is given by

$$f(z) \quad = \quad g \circ M_{C^{-1}}(z) \quad = \quad g\left(\frac{iz+i}{-z+1}\right) \quad = \quad \left(\frac{iz+i}{-z+1}\right)^2.$$

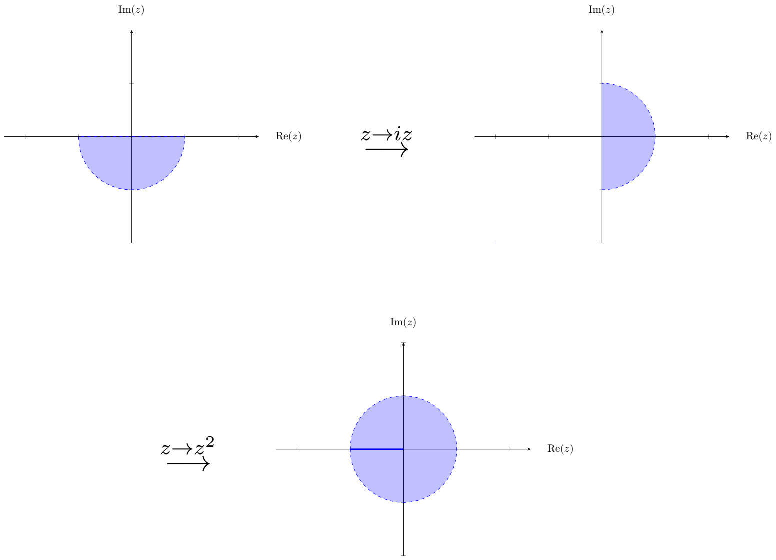


(b) We know from lectures that a holomorphic map f with $f'(z_0) \neq 0$ is conformal at z_0.

We first see that the map $g_1 : z \mapsto \exp(i\pi/2) = iz$ maps $\mathcal{R}_1$ to the right-half of the unit disc $\{z \in \mathbb{C} : |z| < 1, \mathrm{Re}(z) > 0\}$. The map g_1 is conformal on all of $\mathbb{C}$ as it's holomorphic and its derivative does not equal 0.

We next see that the map $g_2 : z \mapsto z^2$ maps $\{z \in \mathbb{C} : |z| < 1, \mathrm{Re}(z) > 0\}$ to $\mathcal{R}_2$. The map g_2 is conformal on $\mathbb{C} \setminus \{0\}$ as $f'(z) = 0 \iff z = 0$. Since $0 \notin \{z \in \mathbb{C} : |z| < 1, \mathrm{Re}(z) > 0\}$, g_2 is conformal on its domain. We thus conclude that the desired map is

$$g(z) = (g_2 \circ g_1)(z) = (iz)^2 = -z^2.$$



(c) The region $\mathcal{R}_2$ can also be described as $\{z \in \mathbb{C} : 0 < |z| < 1, \mathrm{Arg}(z) \in (-\pi, \pi)\}$. Since

$$\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z),$$

we deduce that the principal branch of Log maps $\mathcal{R}_2$ to the horizontal half-strip

$$\{z \in \mathbb{C} : \mathrm{Re}(z) < 0, \mathrm{Im}(z) \in (-\pi, \pi)\}.$$

Q.4 *[Q9, 2004]*

(a) Determine all Möbius transformations T for which $T(\infty) = \infty$ and $T(1) - T(0) = 1$. What is the geometric meaning of T and of its inverse T^{-1}?

(b) Let C_1 be the circle passing through $0, 1, -i$ and C_2 be the circle passing through $0, 1, i$. Let Ω be the intersection of the two discs bounded by C_1 and C_2.

(i) Determine the unique Möbius transformation S which maps the ordered set of points $\{0, 1, -i\}$ to the ordered set of points $\{-1, \infty, i\}$.

(ii) Sketch the image of Ω under S.

S.4 (a) We know that

$$T(z) = \frac{az + b}{cz + d}, \quad \text{for } a, b, c, d \in \mathbb{C}.$$

From the condition $T(\infty) = \infty$ we get that $c = 0$. Consequently,

$$1 = T(1) - T(0) = \frac{a+b}{d} - \frac{b}{d} = \frac{a}{d}$$

which implies that $a = d$. We find that

$$T(z) = z + \frac{b}{a}.$$

If $w = z + \frac{b}{a}$ then $z = w - \frac{b}{a}$ so we deduce that $T^{-1}(z) = z - \frac{b}{a}$. Geometrically T is translation by the complex number $\frac{b}{a}$ and T^{-1} is translation by the complex number $-\frac{b}{a}$.

(b) (i) We know from lectures that a Möbius transformation $w = f(z)$ preserves the cross-ratio, i.e.

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

Thus we need to solve for w in terms of z for our ordered sets $\{z_1, z_2, z_3\} = \{0, 1, -i\}$ and $\{w_1, w_2, w_3\} = \{-1, \infty, i\}$. In this case, we have

$$\begin{aligned}\frac{(-1-i)}{(w-i)} = \frac{(z-1)(0+i)}{(z+i)(0-1)} \quad &\iff \quad \frac{-(1+i)}{(w-i)} = \frac{-i(z-1)}{(z+i)} \\ &\iff \quad i(w-i)(z-1) = (1+i)(z+i) \\ &\iff \quad iw(z-1) = (1+i)(z+i) - z + 1 \\ &\iff \quad iw(z-1) = iz + i \\ &\iff \quad w = \frac{z+1}{z-1}.\end{aligned}$$

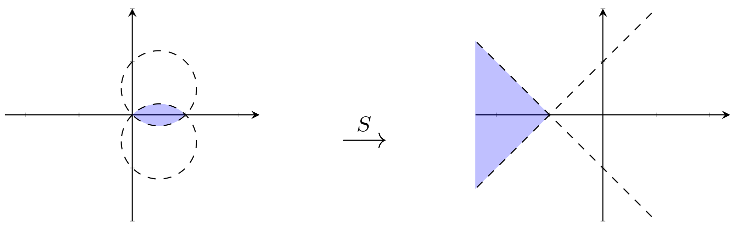
(ii) To find the image of Ω, we first find the image of its boundary. We know from lectures that Möbius transformations map lines and circles to lines and circles. We also know from lectures that 3 points determine a unique line or circle and if one of the points is ∞ then we have a line. Since $0, 1, -i$ lie on C_1, we see that C_1 is mapped to the line ℓ_1 through -1 and i.
Using our formula

$$S(z) = \frac{z+1}{z-1}$$

from above, we have that

$$S(i) = \frac{1-i}{-1-i} = \frac{-2i}{2} = -i.$$

So, by the same reasoning as above, since $0, 1, i$ lie on C_2, we see that C_2 is mapped to the line ℓ_2 through -1 and $-i$.
By continuity of S, the image of Ω must be one of the four connected regions in Figure **??**. To determine the image of Ω, we take $z = \frac{1}{2}$ in the interior and see where it gets mapped to under S, that is $S(\frac{1}{2}) = -3$. We conclude that the image of Ω under S is the region above ℓ_2 and below ℓ_1.



Q.5 *[Q5 2022]*

(a) *Let $U = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ and $V = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that exp is a conformal in $\mathbb{C}$ and satisfies $f(U) = V$. Is exp: $U \to V$ a biholomorphism?*

(b) *Using part (a) or otherwise, find a conformal map from $\{z \in \mathbb{C} : |z| < 1\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$.*

S.5 (a) the function $f(z) = e^z$ is an entire function. Consequently, we know from class that it is conformal on $\mathbb{C}$ at every point where its derivative is not zero. As $f'(z) = e^z \neq 0$ for any $z \in \mathbb{C}$ we conclude that f is conformal on $\mathbb{C}$.
Since $e^z = e^x e^{iy}$ when $z = x+iy$ we see that the line $y = c$, $x < 0$ is mapped to a segment $0 < r < 1$, $\theta = c$. Consequently the image of U by f is the union of all such segments with y ranging over $\mathbb{R}$. This gives us V.
The map is not biholomorphic as it is not even injective – the lines $y = c$, $x < 0$ and $y = c + 2\pi$, $x < 0$, both of which are in U, are mapped to the same segment.

(b) In order to be able to use (a) we need to take $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to U. This will be done by using a Möbius transformation. We know that the Cayley map takes the upper half plane, $\mathbb{H}$, conformally to the unit disc, $\mathbb{D}$. Consequently, its inverse takes $\mathbb{D}$ to $\mathbb{H}$ conformally. To reach U we only need to rotate the domain by $\frac{\pi}{2}$ anti-clockwise, which is again conformal. Formally, we consider the following conformal maps:

$$f_1 : \mathbb{D} \to \mathbb{H}, \qquad f_1(z) = i\frac{z+1}{1-z},$$
$$f_2 : \mathbb{H} \to U, \qquad f_2(z) = e^{\frac{i\pi}{2}} z = iz.$$

We have that $g : \mathbb{D} \to V$ defined by

$$g(z) = f \circ f_2 \circ f_1(z) = e^{\frac{z+1}{z-1}}$$

is conformal and surjective as requested.

Q.6 *[Q2.3 2021]*

Suppose that $t > 0$. Prove that $f_n : \{z \in \mathbb{C} : \operatorname{Re}(z) \geq t\} \to \mathbb{C}$ defined by

$$f_n(z) := \tanh(nz) = \frac{\sinh(nz)}{\cosh(nz)}$$

is uniformly convergent to 1 as $n \to \infty$. Is the convergence uniform in $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$? Justify your answer.

S.6 Let $t > 0$ and let z be such that $\operatorname{Re}(z) \geq t$. We find that

$$f_n(z) = \frac{\sinh(nz)}{\cosh(nz)} = \frac{\exp(nz) - \exp(-nz)}{\exp(nz) + \exp(-nz)} = \frac{1 - \exp(-2nz)}{1 + \exp(-2nz)} = 1 + \frac{-2\exp(-2nz)}{1 + \exp(-2nz)}. \tag{1}$$

Now, by the reverse triangle inequality, we have

$$|f_n(z) - 1| = \left|\frac{-2\exp(-2nz)}{1 + \exp(-2nz)}\right| \leq \frac{2|\exp(-2nz)|}{1 - |\exp(-2nz)|}.$$

Since $\operatorname{Re}(z) \geq t$, we also have that

$$|\exp(-2nz)| = |\exp(-2n\operatorname{Re}(z))| \cdot |\exp(-2ni\operatorname{Im}(z))| = \exp(-2n\operatorname{Re}(z)) \leq \exp(-2nt).$$

Combining the previous two inequalities, we obtain that

$$|f_n(z) - 1| \leq \frac{2\exp(-2nt)}{1 - \exp(-2nt)}.$$

The right-hand side is independent of z and converges to 0 as $n \to \infty$. This proves the uniform convergence of f_n to the constant function $f(z) = 1$ on $\{z \in \mathbb{C} : \operatorname{Re}(z) \geq t\}$.

By the test for (non-)uniform convergence from lectures, we know that if there exists $c > 0$, $\{z_n\} \subset \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ such that for all $n \in \mathbb{N}$, $|f_n(z_n) - f(z_n)| \geq c$ then the convergence is not uniform in the region $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

We could take the sequence $z_n = \frac{1}{n}$. Then, by (1), we have that

$$|f_n(z_n) - f(z_n)| = \frac{2\exp(-2)}{1 + \exp(-2)} =: c > 0$$

which proves that the convergence is not uniform in $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

Q.7 *[Q3.2 2020 Resit]*

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{3^n + z^n}$$

is uniformly convergent on $|z| \leq \rho$ for every real ρ with $0 < \rho < 3$. Is the convergence uniform on $|z| < 3$? [Hint (which was not provided in the original exam): You may use without proof the fact that if $f_n(z)$ doesn't converge uniformly to zero on a set U, then the series $\sum_{n=1}^{\infty} f_n(z)$ can't converge uniformly]

S.7 We apply the Weierstrass M-test. Let $|z| \leq \rho < 3$. By reverse triangle inequality,

$$|3^n + z^n| \geq 3^n - |z|^n \geq 3^n - \rho^n.$$

Denoting by $M_n := \frac{1}{3^n - \rho^n}$ we find that

$$\frac{1}{|3^n + z^n|} \leq M_n$$

for all z with $|z| \leq \rho$. We claim that

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{3^n - \rho^n}$$

is convergent by the ratio test. Indeed,

$$\frac{M_{n+1}}{M_n} = \frac{3^n - \rho^n}{3^{n+1} - \rho^{n+1}} \underset{n\to\infty}{\longrightarrow} \frac{1}{3} < 1.$$

Hence the conditions for Weierstrass' M-test are satisfied and the series converges uniformly on $|z| \leq \rho$.

We will show that the convergence is not uniform on $|z| < 3$ by utilising the hint. By the test for (non-)uniform convergence from lectures, we need to find $c > 0$ and $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ with $|z_N| < 3$ such that for all $N \in \mathbb{N}$,

$$\left| \frac{1}{3^n + z_n^n} - 0 \right| \geq c.$$

Choosing $z_n = -\left(3^n - 1\right)^{\frac{1}{n}}$ we find that

$$|z_n| = (3^n - 1)^{\frac{1}{n}} < (3^n)^{\frac{1}{n}} = 3$$

and

$$\left| \frac{1}{3^n + z_n^n} - 0 \right| = 1.$$

This proves that the convergence is not uniform on $|z| < 3$.

Q.8 *[Q3.3 2021]*

Let p be a polynomial with complex coefficients. Using a parametrisation of the unit circle, show that

$$\overline{p'(0)} = \frac{1}{2\pi i} \int_{|z|=1} \overline{p(z)}\, dz.$$

Find $\int_{|z|=1} \mathrm{Re}(p(z))\, dz$.

S.8 We write $p(z) = \sum_{k=0}^{n} a_k z^k$. We parametrise the unit circle as $\gamma(t) = \exp(it)$ where $t \in [0, 2\pi]$. Then we have that

$$\begin{aligned}\int_\gamma \overline{p(z)}\, dz &= \int_\gamma \sum_{k=0}^{n} \overline{a_k}\,\overline{z^k}\, dz = \int_0^{2\pi} \sum_{k=0}^{n} \overline{a_k}\,\overline{\gamma(t)^k} \cdot \gamma'(t)\, dt \\ &= \sum_{k=0}^{n} \overline{a_k} \int_0^{2\pi} \exp(-ikt) \cdot i \exp(it)\, dt = \sum_{k=0}^{n} i\overline{a_k} \int_0^{2\pi} \exp((1-k)it)\, dt.\end{aligned}$$

The integral $\int_0^{2\pi} \exp((1-k)it)\, dt$ is equal to zero unless $1 - k = 0$, that is $k = 1$, in which case it is equal to 2π. Hence

$$\int_\gamma \overline{p(z)}\, dz = 2\pi i \overline{a_1} = 2\pi i \overline{p'(0)}$$

from which we obtain the desired result.

For the second part of the question, we use that $\mathrm{Re}(p(z)) = \frac{p(z)+\overline{p(z)}}{2}$ to obtain

$$\int_{|z|=1} \mathrm{Re}(p(z))\,dz = \frac{1}{2}\int_{|z|=1} p(z)\,dz + \frac{1}{2}\int_{|z|=1} \overline{p(z)}\,dz.$$

Since p is a polynomial, it has a holomorphic antiderivative and the contour is closed so $\int_{|z|=1} p(z)\,dz = 0$ by the Complex Fundamental Theorem of Calculus (Theorem 6.10). Hence

$$\int_{|z|=1} \mathrm{Re}(p(z))\,dz = \frac{1}{2}\int_{|z|=1} \overline{p(z)}\,dz = \pi i\overline{p'(0)}.$$

Q.9 (a) *[Q9a 2012]*

Show that there exists an unbounded open subset $S \subset \mathbb{C}$ on which $\sin(z)$ is bounded.

(b) *[Q6 2011]*

Show that there exists no holomorphic function f such that $f(z) = |\sin(z)|$ for all purely real $z = x$ with $-1 < x < 1$.

S.9 (a) By the triangle inequality that

$$|\sin(z)| = \left|\frac{e^{iz} - e^{-iz}}{2i}\right| \leq \frac{|e^{iz}|}{2} + \frac{|e^{-iz}|}{2}.$$

Consider the horizontal strip $D = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < 1\}$, which is open and unbounded. We claim $\sin(z)$ is bounded on D. To see this, let $z \in D$ and write $z = x + iy$ for $x, y, \in \mathbb{R}$. We have $|e^{iz}| = |e^{ix-y}| = e^{-y} < e$, since $y = \mathrm{Im}(z) > -1$. Moreover, $|e^{-iz}| = |e^{y-ix}| = e^{y} < e$, since $y = \mathrm{Im}(z) < 1$. Hence $|\sin(z)| < \frac{e}{2} + \frac{e}{2} = e$ when $z \in D$ as claimed.

(b) Suppose there is a holomorphic function f on $D = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(x) = |\sin(x)|$ for $-1 < x < 1$. Then $f(x) = \sin(x)$ for $0 \leq x < 1$ and $f(x) = -\sin(x)$ for $-1 < x < 0$. Let $I = [0, 1) \subset D$. Then I is contained in the set of points $S = \{z \in \mathbb{C} :\ f(z) = \sin(z)\}$. In particular, the set S contains a non-isolated point. By the identity theorem we must have that $f(z) = \sin(z)$ on all of D, since $\sin(z)$ is holomorphic on D. Similarly, if we take $I = (-1, 0]$ we get $f(z) = -\sin(z)$ on D, a contradiction.

Q.10 *[Q1.4 2020]*

Let f be a holomorphic function on $\mathbb{C} - \{0\}$. Show that f is bounded if and only if f is constant. State clearly any results you use from lectures.

S.10 Clearly if f is constant then it is bounded. So, let us assume f is bounded and show it is constant.

If f is bounded on $\mathbb{C} - \{0\}$ by some constant $M > 0$, it is in particular bounded on $B_1^*(0) = B_1(0) - \{0\}$. By the Riemann Extension Theorem f must have a removable singularity at $z = 0$ (or to apply its proof directly, $0 < |zf(z)| < M|z|$ so by squeezing $\lim_{z\to 0} zf(z) = 0$, and we know this is equiv to having a removable singularity). It follows that f extends to a holomorphic function g defined at $z = 0$. But then g is entire and bounded, so by Liouville's Theorem it is constant (and so must f be).

Q.11 *[Q3.1 2022 Resit]*

Consider the meromorphic function $f(z) = \dfrac{1}{z^2(8 + z^3)}$.

Determine the Laurent series expansion of $f(z)$ on the annulus $\mathcal{A} = \{z \in \mathbb{C} :\ 0 < |z| < 2\}$.

S.11 We use the known geometric series Taylor expansion

$$\frac{1}{1-\omega} = \sum_{k=0}^{\infty} \omega^k, \qquad |\omega| < 1. \tag{2}$$

On the annulus $A = \{z \in \mathbb{C} : 0 < |z| < 2\}$, we have that $|(-z/2)^3| = (|z|/2)^3 < 1$, so by the above expansion (2),

$$\frac{1}{8+z^3} = \frac{1}{8} \cdot \frac{1}{1-(-\frac{z}{2})^3} = \frac{1}{8}\sum_{n=0}^{\infty}\left(\left(-\frac{z}{2}\right)^3\right)^n = \frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n}}z^{3n}.$$

By the uniqueness of Laurent series expansions, it follows that on A we have

$$f(z) = \frac{1}{z^2}\cdot\frac{1}{8+z^3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n+3}}z^{3n-2}.$$

Q.12 *[Q4 2019]*

(i) *Find all the zeros and poles, with their orders, of* $f(z) = \dfrac{z}{\sin z + \cos z}$.

(ii) *Find the residue of f at each of its poles.*

S.12 (i) There is a simple zero (ie, a zero of order 1) at $z = 0$ (because the denominator is not zero).

The poles occur when $\sin z + \cos z = 0$. This is equivalent to $e^{iz} - e^{iz} + i(e^{iz} + e^{-iz}) = 0$; or $e^{2iz}(1+i) = 1-i$, i.e., $e^{2iz} = (1-i)/(1+i) = -i = e^{i3\pi/2}$, i.e., $e^{i(2z-3\pi/2)} = 1$, i.e., $2z - 3\pi/2 = 2\pi n$, i.e., $z = \pi(n+3/4)$, where n is an integer.

The derivative of the denominator, which is $\cos z - \sin z$, is equal to $-(-1)^n\sqrt{2}$ at these poles. Thus these are simple poles.

(ii) Since we have simple poles we can use Rule 3 from lectures (differentiate the denominator) to calculate that the residue at each of $z = \pi(n+3/4)$ is

$$-\frac{\pi(n+3/4)}{(-1)^n\sqrt{2}} = \frac{(-1)^n\pi(4n+3)}{4\sqrt{2}}.$$

Q.13 *[Q9 2016]*

Consider the function $g(z) = \dfrac{e^{-z^2}}{1+e^{-2az}}$, *where* $a = (1+i)\frac{\sqrt{\pi}}{\sqrt{2}} = e^{i\pi/4}\sqrt{\pi}$ *is fixed.*

(a) *Show* $a^2 = i\pi$ *and* $e^{-2a(z+a)} = e^{-2az}$. *Use this to show that*

$$g(z) - g(z+a) = e^{-z^2}. \qquad (*)$$

(b) *Show that all poles of g occur at $z = \frac{a}{2} + na$ with $n \in \mathbb{Z}$. Compute the residue at $z = \frac{a}{2}$.*

(c) *For r and s positive real numbers, consider the contour γ given by the boundary of the parallelogram with vertices $s, s+a, -r+a$ and $-r$. Draw the contour marking all the poles of $g(z)$.*

(d) *Use $(*)$ to show that the horizontal line integrals of $\int_\gamma g(z)dz$ combine to $\int_{-r}^{s} e^{-x^2}dx$. Use this and Cauchy's residue theorem to find an expression for $\int_{-r}^{s} e^{-x^2}dx$.*

(e) *Conclude*

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

S.13 Consider the function $g(z) = \frac{e^{-z^2}}{1+e^{-2az}}$, where $a = (1+i)\frac{\sqrt{\pi}}{\sqrt{2}} = e^{i\pi/4}\sqrt{\pi}$ is fixed.

(a) It is obvious that $a^2 = i\pi$ and hence $e^{-2a^2} = e^{-2\pi i} = 1$ and $e^{-2a(z+a)} = e^{-2az}e^{-2a^2} = e^{-2az}$. Finally,

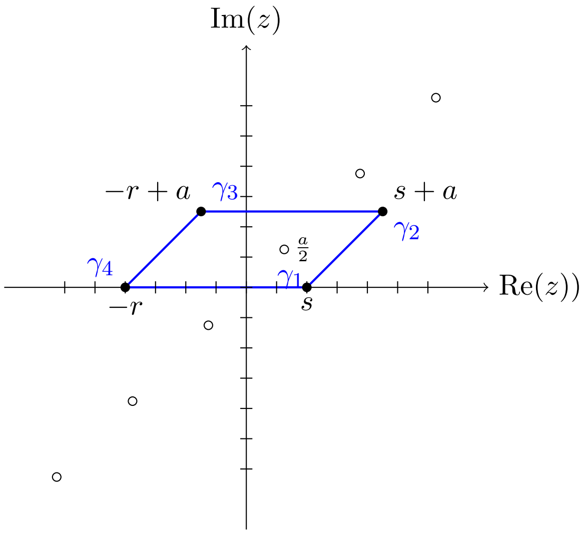
$$g(z) - g(z+a) = \frac{e^{-z^2}}{1+e^{-2az}} - \frac{e^{-(z+a)^2}}{1+e^{-2a(z+a)}} = \frac{e^{-z^2}}{1+e^{-2az}} - \frac{e^{-z^2}e^{-2az}}{1+e^{-2az}} = \frac{e^{-z^2}(1+e^{-2az})}{(1+e^{-2az})},$$

which implies $g(z) - g(z+a) = e^{-z^2}$.

(b) Poles occur when $e^{-2az} = -1$, so when $2az = i\pi + 2\pi i n$ for some $n \in \mathbb{Z}$. Hence by (a) we have $z = (i\pi + 2\pi i n)/(2a) = (ai\pi + 2a\pi i n)/(2a^2) = a/2 + an$. These are all simple poles. Evaluating the residue at $z = a/2$ using Rule 3 we get

$$\mathrm{Res}_{z=a/2}(g(z)) = \left.\frac{e^{-z^2}}{\frac{d}{dz}(1+e^{-2az})}\right|_{z=a/2} = \frac{e^{-(a/2)^2}}{-2ae^{-a^2}} = \frac{e^{-i\pi/4}}{-2ae^{-i\pi}} = -\frac{i}{2\sqrt{\pi}}.$$

(c) Below is a sketch:



(d) Write γ for the contour and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ for the line segments. By Cauchy's residue theorem we have

$$\int_{\gamma_1} g + \int_{\gamma_2} g + \int_{\gamma_3} g + \int_{\gamma_4} g = \int_\gamma g(z)dz = 2\pi i \cdot \mathrm{Res}_{z=a/2}(g(z)) = 2\pi i\left(-\frac{i}{2\sqrt{\pi}}\right) = \sqrt{\pi}.$$

Next note that $\gamma_1 = -\gamma_3 + a$ so by part (a) we have

$$\int_{\gamma_1} g(z)dz + \int_{\gamma_3} g(z)dz = \int_{-r}^{s} g(x)dx - \int_{-r}^{s} g(x+a)dx = \int_{-r}^{s} e^{-x^2}dx.$$

So

$$\int_{-r}^{s} e^{-x^2}dx = \sqrt{\pi} - \int_{\gamma_2} g(z)dz - \int_{\gamma_4} g(z)dz.$$

(e) Finally, we need to show the latter integrals relating to γ_2 and γ_4 vanish as s and r tend to infinity respectively. By the Estimation Lemma we have (for $i = 2, 4$) that

$$\left|\int_{\gamma_i} g(z)dz\right| \le |a| \sup_{z\in\gamma_i}\left|\frac{e^{-z^2}}{1+e^{-2az}}\right| \le \sqrt{\pi}\sup_{z\in\gamma_i}\frac{|e^{-x^2+y^2-2xyi}|}{|1-|e^{-2a(x+iy)}||} = \sup_{z\in\gamma_i}\frac{e^{-x^2+y^2}}{|1-e^{\sqrt{2\pi}(y-x)}|}.$$

Note that on both contours $0 \le y \le \mathrm{Im}(a) = \frac{\sqrt{\pi}}{\sqrt{2}}$. Thus, the numerator above is bounded above by $\sqrt{\pi}e^{\pi/2}\sup_{z\in\gamma_i} e^{-x^2}$. Furthermore, for γ_2 (where $x > 0$) the denominator is bounded below by $1 - e^{\pi/\sqrt{2}}e^{-\sqrt{2\pi}x}$. For γ_4 (where $x < 0$) the denominator is bounded below by $e^{-\sqrt{2\pi}x} - 1$. Since on γ_2 we have $x \to \infty$ as $s \to \infty$, and on γ_4 we have $x \to -\infty$ as $r \to \infty$, in both cases our estimate tends to zero as s and r tend to infinity respectively.

Q.14 (*a*) *[Q1.4 2020]*

Show that the polynomial $z^5 + 15z + 1$ *has precisely four zeros (counted with multiplicity) in the set* $\{\, z : \frac{3}{2} \le |z| < 2\}$.

(*b*) *[Q5b 2018]*

Fix $R > 0$. Prove that if N is sufficiently large, depending on R, then $\displaystyle\sum_{k=0}^{N} \frac{z^k}{k!} = 0$ has no solutions $z \in D(0, R)$. You can use any properties of the exponential function that you like, provided they are stated clearly.

S.14 (a) We apply Rouché's Theorem twice, for different choices of functions.

First let $f(z) = z^5 + 15z + 1$ and $g(z) = z^5$. For $|z| = 2$ we have

$$|f(z) - g(z)| = |15z + 1| \leq 1 + 30 < 32 = |g(z)|.$$

Hence the 5 zeros of $z^5 + 15z + 1$ are all in $\{z : |z| < 2\}$.

Next, let $f(z) = z^5 + 15z + 1$ and $g(z) = 15z$. Then for $|z| = 3/2$ we have

$$|f(z) - g(z)| = |z^5 + 1| \leq 3^5/2^5 + 1 = 243/32 + 1 < 12 < 45/2 = |g(z)|.$$

Since $g(z)$ has one zero with $|z| < 3/2$, so does f. Taking the difference, f has 4 zeros in the region given in the question.

(b) Let $p_N(z)$ be the partial sum appearing in the question. We apply Rouché's Theorem with $g(z) = e^z$, $f(z) = p_N(z)$, and write γ_R for the circular contour of radius R centred at 0. For $z = x + iy \in \gamma_R$ we have $|g(x + iy)| = |e^x e^{iz}| = e^x \geq e^{-R}$. Since p_N is the nth partial sum of the power series expansion of e^z at $z = 0$, and e^z is holomorphic on $\mathbb{C}$, we have $p_N(z) \to e^z$ as $N \to \infty$ uniformly on any compact set (by results from term 1). In particular there is M depending on R such that for $N > M$ we have $|f(z) - g(z)| \leq \frac{e^{-R}}{2} < e^{-R} \leq |g(z)|$ for all $z \in \gamma_R$. Since $p_N(z)$ and e^z are holomorphic on all of $\mathbb{C}$, the hypotheses for Rouché's Theorem are satisfied for $N > M$. Therefore for $N > M$, the functions $p_N(z)$ and e^z have the same number of zeros in $D(0, R)$, and we know the exponential function has no zeros in $\mathbb{C}$.