Q.1 *[Q1 from the May 2023 exam]*

1.1 *Let $U \subset \mathbb{C}$ be an open set. Define what it means for a function $f : U \to \mathbb{C}$ to be complex differentiable at a point $z_0 \in U$.*

1.2 *State the Cauchy-Riemann equations.*

1.3 *Let $f : U \to \mathbb{C}$ be the function defined by*

$$f(z) = f(x + iy) = x\cos(y) + \sinh(iy)\cosh(x).$$

Use the Cauchy-Riemann equations to determine the points $z_0 \in \mathbb{C}$ where f is complex differentiable.

Q.2 *[Q2 from the May 2023 exam]*

2.1.(a) *On what subset of $\mathbb{C}$ is the function $f(z) = (z + i)^4 - 3$ conformal? Justify your response.*

2.1.(b) *Describe the geometric effects of $f(z)$ on the tangent vectors of the curves passing through the point $z = 1 - 2i$.*

2.2 *Let $\gamma : [0, 3] \to \mathbb{C}$ be the contour given by*

$$\gamma(t) := \begin{cases} 2t, & \text{if } 0 \le t \le 1, \\ 4 - 2i + 2(-1 + i)\,t, & \text{if } 1 \le t \le 2, \\ 2(3 - t)\,i, & \text{if } 2 \le t \le 3. \end{cases}$$

(a) Sketch $\gamma(t)$ in $\mathbb{C}$.
(b) Evaluate $\int_\gamma \cos(z)dz$.

Q.3 *[Q5 from the May 2023 exam]*

3.1 *Prove that for each $a \in \mathbb{R}$, $a > 0$, the series*

$$\sum_{n=1}^{\infty} n^{-z}$$

converges uniformly on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 1 + a\}$ where n^{-z} is defined using the principal logarithm [You may use without proof that $\sum_{n=1}^{\infty} n^{-b}$, $b \in \mathbb{R}$, $b > 1$, converges.]

3.2 *Does the series $\sum_{n=1}^{\infty} n^{-z}$ defines a continuous function on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 1\}$? Justify your response.*

3.3 *Does the series $\sum_{n=1}^{\infty} n^{-z}$ converge uniformly on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \ge 1\}$? Justify your response.*

Q.4 *[Q6 from the May 2023 exam] Consider the set $U = \mathbb{C} \setminus \{iy \ : \ y \in \mathbb{R},\ y \le 0\}$.*

4.1 *Sketch the set U in $\mathbb{C}$.*

4.2 *Is U an open set? Justify your response.*

4.3 *Find a biholomorphic map from U to the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ and justify why this map is biholomorphic.*