Q.1 *[Q1 from the May 2023 exam]*

1.1 *Let* $U \subset \mathbb{C}$ *be an open set. Define what it means for a function* $f : U \to \mathbb{C}$ *to be complex differentiable at a point* $z_0 \in U$.

1.2 *State the Cauchy-Riemann equations.*

1.3 *Let* $f : U \to \mathbb{C}$ *be the function defined by*

$$f(z) = f(x + iy) = x\cos(y) + \sinh(iy)\cosh(x).$$

Use the Cauchy-Riemann equations to determine the points $z_0 \in \mathbb{C}$ *where* f *is complex differentiable.*

Q.2 *[Q2 from the May 2023 exam]*

2.1.(a) *On what subset of* $\mathbb{C}$ *is the function* $f(z) = (z+i)^4 - 3$ *conformal? Justify your response.*

2.1.(b) *Describe the geometric effects of* $f(z)$ *on the tangent vectors of the curves passing through the point* $z = 1 - 2i$.

2.2 *Let* $\gamma : [0,3] \to \mathbb{C}$ *be the contour given by*

$$\gamma(t) := \begin{cases} 2t, & \textit{if } 0 \le t \le 1, \\ 4 - 2i + 2\,(-1+i)\,t, & \textit{if } 1 \le t \le 2, \\ 2\,(3-t)\,i, & \textit{if } 2 \le t \le 3. \end{cases}$$

(a) Sketch $\gamma(t)$ *in* $\mathbb{C}$.
(b) Evaluate $\int_\gamma \cos(z) dz$.

Q.3 *[Q5 from the May 2023 exam]*

3.1 *Prove that for each* $a \in \mathbb{R}$, $a > 0$, *the series*

$$\sum_{n=1}^{\infty} n^{-z}$$

converges uniformly on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 1 + a\}$ *where* n^{-z} *is defined using the principal logarithm [You may use without proof that* $\sum_{n=1}^{\infty} n^{-b}$, $b \in \mathbb{R}$, $b > 1$, *converges.]*

3.2 *Does the series* $\sum_{n=1}^{\infty} n^{-z}$ *defines a continuous function on* $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 1\}$*? Justify your response.*

3.3 *Does the series* $\sum_{n=1}^{\infty} n^{-z}$ *converge uniformly on* $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \geq 1\}$*? Justify your response.*

Q.4 *[Q6 from the May 2023 exam] Consider the set* $U = \mathbb{C} \setminus \{iy \,:\, y \in \mathbb{R},\ y \leq 0\}$.

4.1 Sketch the set U *in* $\mathbb{C}$.

4.2 Is U *an open set? Justify your response.*

4.3 Find a biholomorphic map from U *to the open unit disc* $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ *and justify why this map is biholomorphic.*