Q.1 *[Q1 from the May 2023 exam]*

1.1 Let $U \subset \mathbb{C}$ be an open set. Define what it means for a function $f : U \to \mathbb{C}$ to be complex differentiable at a point $z_0 \in U$.

1.2 State the Cauchy-Riemann equations.

1.3 Let $f : U \to \mathbb{C}$ be the function defined by

$$f(z) = f(x+iy) = x\cos(y) + \sinh(iy)\cosh(x).$$

Use the Cauchy-Riemann equations to determine the points $z_0 \in \mathbb{C}$ where f is complex differentiable.

Sol:

1.1) A function $f : U \to \mathbb{C}$ is complex diff. at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

1.2) $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann eq. at (x_0, y_0) if

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

1.3) We start by writing f as $u + iv$. To do so we need ~~to~~ to simplify $\sinh(iy)$. By def:

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

and as such

$$\sinh(iy) = \frac{e^{iy} - e^{-iy}}{2} = i\sin(y).$$

We conclude that

$$f(x+iy) = \underbrace{x\cos(y)}_{u(x,y)} + i\underbrace{\sin(y)\cosh(x)}_{v(x,y)}$$

$u, v \in C^1(\mathbb{R}^2)$ so we know from class that f is complex diff at $z_0 = x_0 + iy_0$ if and only if C-R eq. hold at (x_0, y_0).

$u_x(x,y) = \cos y$ $\quad$ $u_y(x,y) = -x\sin(y)$

$v_x(x,y) = \sin(y)\sinh(x)$ $\quad$ $v_y(x,y) = \cos(y)\cosh(x)$

C-R eq. are:

(1) $\cos(y) = \cos(y)\cosh(x)$ $\quad$ (2) $x\sin(y) = \sin(y)\sinh(x)$

(1) implies $\cos(y) = 0$ or $1 = \cosh(x)$

$$\Downarrow \qquad\qquad \Downarrow$$

$$y = \tfrac{\pi}{2} + \pi k \quad k \in \mathbb{Z} \qquad x = 0$$

<u>if $y = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$</u>, $\sin(y) = (-1)^k$ and as such (2) becomes $x = \sinh(x)$, i.e $x = 0$.

<u>if $x = 0$</u> (2) is immediate.

$\Rightarrow$ C-R holds for $x = 0$, $y \in \mathbb{R}$. Therefor f is complex diff on $i\mathbb{R}$.

Q.2 *[Q2 from the May 2023 exam]*

2.1.(a) *On what subset of $\mathbb{C}$ is the function $f(z) = (z+i)^4 - 3$ conformal? Justify your response.*

2.1.(b) *Describe the geometric effects of $f(z)$ on the tangent vectors of the curves passing through the point $z = 1 - 2i$.*

2.2 *Let $\gamma : [0,3] \to \mathbb{C}$ be the contour given by*

$$\gamma(t) := \begin{cases} 2t, & \text{if } 0 \le t \le 1, \\ 4 - 2i + 2(-1+i)t, & \text{if } 1 \le t \le 2, \\ 2(3-t)i, & \text{if } 2 \le t \le 3. \end{cases}$$

(a) Sketch $\gamma(t)$ in $\mathbb{C}$.
(b) Evaluate $\int_\gamma \cos(z)dz$.

Sol:

(2.1)(a) From class we know that if f is holomorphic in a domain D then f is conformal at $z_0 \in D$ if and only if $f'(z_0) \neq 0$.

Our function is entire and as such f is conformal at z_0 if and only if $f'(z_0) \neq 0$.

$$f'(z) = 4(z+i)^3$$

$\Rightarrow f$ is conformal in $\mathbb{C} \setminus \{-i\}$.

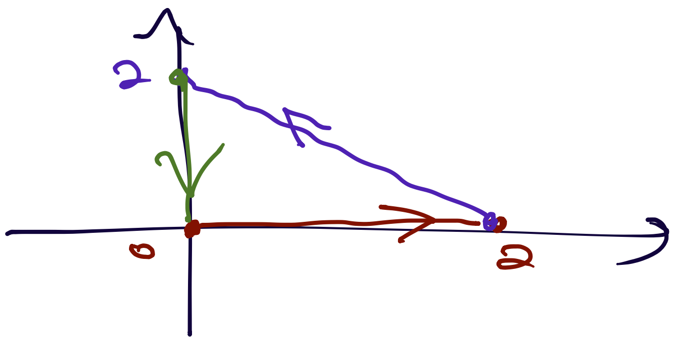
(b) We saw in class that a tangent to a curve passing at z_0 is multiplied by $f'(z_0)$ in its image.

$$f'(1-2i) = 4(1-2i+i)^3 = 4(1-i)^3 = 4\left(\sqrt{2}\, e^{-i\frac{\pi}{4}}\right)^3$$

$$= 4 \cdot 2^{3/2} \cdot e^{-i\frac{3\pi}{4}} = 2^{7/2} e^{-i\frac{3\pi}{4}}.$$

The tangent vector gets stretched by $2^{7/2}$ and is rotated $3\pi/4$ clockwise.

(22) (a)



$0 \le t \le 1$ $\quad x(t) = 2t$ $\quad y(t) = 0$

$1 \le t \le 2$ $\quad x(t) = 4 - 2t$ $\quad y(t) = -2 + 2t$

$2 \le t \le 3$ $\quad x(t) = 0$ $\quad y(t) = 2(3 - t)$

(b)

$$\int_\gamma \cos(z)\, dz = 0$$

γ is a closed contour and

$$\cos z = \frac{d}{dz}(-\sin z)$$

so the result follows from FTC.

Q.3 *[Q5 from the May 2023 exam]*

3.1 *Prove that for each $a \in \mathbb{R}$, $a > 0$, the series*

$$\sum_{n=1}^{\infty} n^{-z}$$

converges uniformly on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 1 + a\}$ where n^{-z} is defined using the principal logarithm [You may use without proof that $\sum_{n=1}^{\infty} n^{-b}$, $b \in \mathbb{R}$, $b > 1$, converges.]

3.2 *Does the series $\sum_{n=1}^{\infty} n^{-z}$ defines a continuous function on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 1\}$? Justify your response.*

3.3 *Does the series $\sum_{n=1}^{\infty} n^{-z}$ converge uniformly on $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \geq 1\}$? Justify your response.*

Sol: (a) $f_n(z) = n^{-z}$

Real function

$$|n^{-z}| = |e^{-z\,\mathrm{Log}\,n}| = |e^{-z(\log n + i\,\mathrm{Arg}(n))}|$$

$$= |e^{-\log n(x+iy)}| = |e^{-x\log(n)} \cdot e^{-iy\log n}|$$

$$= e^{-x\log(n)} = n^{-x} = n^{-\mathrm{Re}(z)}$$

For $z \in \{\mathrm{Re}(z) > 1+a\}$ we get

$$|f_n(z)| = n^{-\mathrm{Re}(z)} \leq n^{-(1+a)} := M_n$$

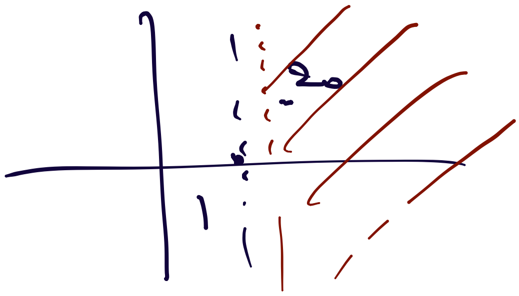
By the hint

$$\sum_{n=1}^{\infty} M_n < \infty$$

Consequently by Weierstrass' M-test the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly.

(b) We know that $f_n(z)$ are cont. on $\{\mathrm{Re}(z) > 1\}$ and as such if $\sum_{n=1}^{\infty} f_n(z)$ converges locally uniformly on $\{\mathrm{Re}(z) > 1\}$ the limit will be cont.

Indeed let $z_0 \in \{Re(z) > 1\}$



The set

$$D_{z_0} = \left\{ w \in \mathbb{C} \;\middle|\; Re(w) > \frac{1+Re(z_0)}{2} \right\}$$

is open and $\frac{1+Re(z_0)}{2} > 1$

so we can write

$$\frac{1+Re(z_0)}{2} = 1 + a_{z_0} \qquad a_{z_0} > 0$$

and use (a) to see that $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly in D_{z_0}.

(c) As $\sum n^{-1} = \infty$

the series can't converge on $\{Re(z) \geq 1\}$.

Q.4 *[Q6 from the May 2023 exam] Consider the set* $U = \mathbb{C} \setminus \{iy \;:\; y \in \mathbb{R},\ y \leq 0\}$.

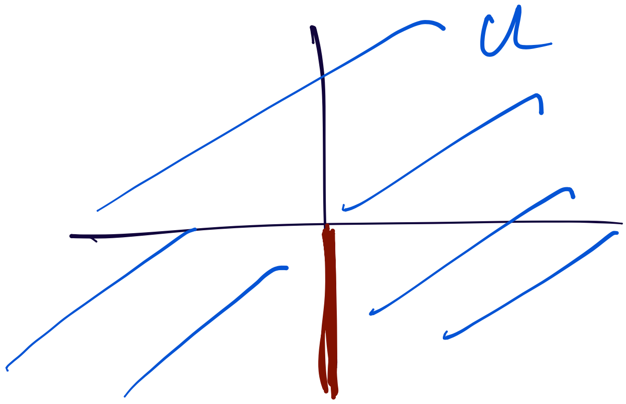
4.1 *Sketch the set* U *in* $\mathbb{C}$.

4.2 *Is* U *an open set? Justify your response.*

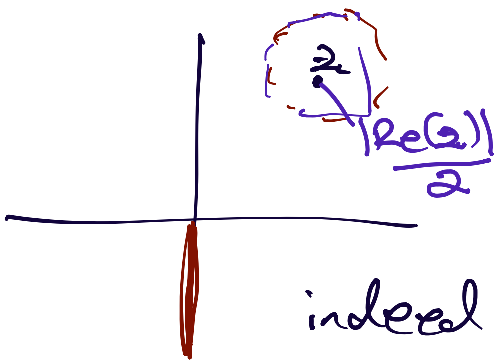
4.3 *Find a biholomorphic map from* U *to the open unit disc* $\mathbb{D} = \{z \in \mathbb{C} \;:\; |z| < 1\}$ *and justify why this map is biholomorphic.*

Sol:

(a)



(b) U is open



if $z \in U$ and $\mathrm{Re}(z) \neq 0$

$B_{\frac{|\mathrm{Re}(z)|}{2}}(z) \subset U$

indeed, if $w \in B_{\frac{|\mathrm{Re}(z_0)|}{2}}(z_0)$

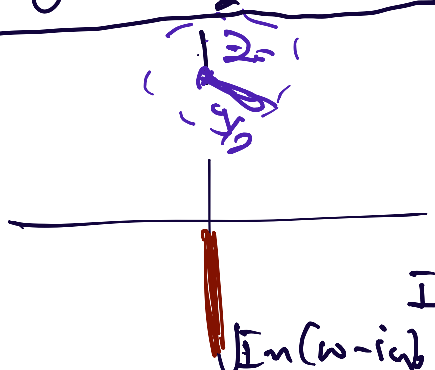
then

$$\mathrm{Re}(w) = \mathrm{Re}(z_0) + \mathrm{Re}(w - z_0)$$

$|\mathrm{Re}(w - z_0)| \leq |w - z_0| < \frac{|\mathrm{Re}(z_0)|}{2}$ and as such

$$|\mathrm{Re}(w)| \geq \Big|\,|\mathrm{Re}(z_0)| - \underbrace{|\mathrm{Re}(w - z_0)|}_{< \frac{|\mathrm{Re}(z_0)|}{2}}\Big| > \frac{|\mathrm{Re}(z_0)|}{2} > 0$$

$\to w \in U$. As w was arbitrary $B_{\frac{|\mathrm{Re}(z)|}{2}}(z) \subset U$



if $z_0 \in U$ and $z_0 = iy_0$ $y_0 > 0$

$B_{\frac{y_0}{2}}(z_0) \subset U$

indeed, if $w \in B_{\frac{y_0}{2}}(iy_0)$ then

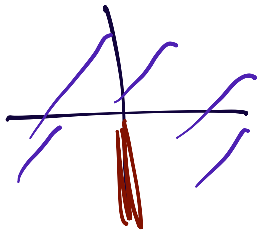
$\mathrm{Im}(w) = y_0 + \mathrm{Im}(w - iy_0)$

$|\mathrm{Im}(w - iy_0)| < \frac{y_0}{2}$ and as such

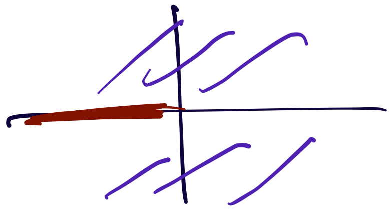
$\mathrm{Im}(w) > y_0 - |\mathrm{Im}(w - iy_0)| > \frac{y_0}{2} > 0$. This implies that $w \in U$

As w was arbitrary, $B_{y_0/2}(iy_0) \subset \mathcal{U}$.

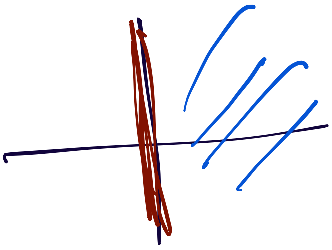
(c)



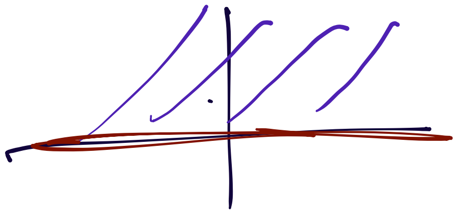
$\xrightarrow{e^{-i\pi/2} z}$



$\xrightarrow{\sqrt{z}}$

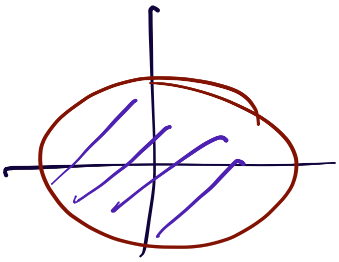


$\xrightarrow{e^{i\pi/2} z}$



$\xrightarrow{M_c(z) = \frac{z-i}{z+i}}$

Cayley map



All maps have holo. inverse in the appropriate domain.

- The inverse of $e^{-i\pi/2} z$ is $e^{i\pi/2} z$ (and vice versa)
- M_c is a Möbius and as such biholomorphic
- The principal branch of $\sqrt{z}$ in $\mathbb{C} \setminus \{\operatorname{Re} z \leq 0\}$ is holo. with a holo. inverse z^2.

The composition of these maps is biholo.

and gives the desired map.