Q.1 *[Q1 from the May 2023 exam]*

1.1 Let $U \subset \mathbb{C}$ be an open set. Define what it means for a function $f : U \to \mathbb{C}$ to be complex differentiable at a point $z_0 \in U$.

1.2 State the Cauchy-Riemann equations.

1.3 Let $f : U \to \mathbb{C}$ be the function defined by

$$f(z) = f(x + iy) = x\cos(y) + \sinh(iy)\cosh(x).$$

Use the Cauchy-Riemann equations to determine the points $z_0 \in \mathbb{C}$ where f is complex differentiable.

S.1

1.1 A function $f : U \to \mathbb{C}$ is complex differentiable at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

1.2 The pair of functions $u, v : U \to \mathbb{R}$ satisfy the Cauchy-Riemann equation at $z_0 = x_0 + iy_0$ if

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

1.3 We know that a function $f = u+iv$, with $u, v : U \to \mathbb{R}$, is complex differentiable at $z_0 = x_0+iy_0 \in U$ if and only if u and v are differentiable at (x_0, y_0) and satisfy the Cauchy-Riemann equations. In order to identify u and v we first need to simplify $\sinh(iy)$. We have that

$$\sinh(iy) = \frac{e^{iy} - e^{-iy}}{2} = i\sin(y)$$

and consequently

$$f(z) = x\cos(y) + i\sin(y)\cosh(x) = u(x, y) + iv(x, y)$$

where $u(x, y) = x\cos(y)$ and $v(x, y) = \sin(y)\cosh(x)$. We have that

$$u_x(x, y) = \cos(y), \qquad v_y(x, y) = \cos(y)\cosh(x),$$

$$u_y(x, y) = -x\sin(y), \qquad v_x(x, y) = \sin(y)\sinh(x).$$

Using Cauchy-Riemann equation we find that

$$\cos(y) = \cos(y)\cosh(x), \qquad x\sin(y) = \sin(y)\sinh(x).$$

The first equation implies that $\cos(y) = 0$ or $\cosh(x) = 1$, i.e. $y = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$ or $x = 0$.
<u>When $y = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$:</u> we have that the second equation reads as

$$(-1)^n x = x\sin\left(\frac{\pi}{2} + n\pi\right) = \sin\left(\frac{\pi}{2} + n\pi\right)\sinh(x) = (-1)^n \sinh(x)$$

or $x = \sinh(x)$. There is only one solution to this equation, which is $x = 0$.
<u>When $x = 0$:</u> we have that the second equation reads as

$$0 = 0 \cdot \sin(y) = \sin(y)\sinh(0) = 0,$$

i.e. the equation always holds.
In conclusion, Cauchy-Riemann equations hold for any (x_0, y_0) such that $x_0 = 0$ and as u and v are differentiable we conclude that f is complex differentiable if and only if $z_0 \in i\mathbb{R}$.

Q.2 *[Q2 from the May 2023 exam]*

2.1.(a) *On what subset of $\mathbb{C}$ is the function $f(z) = (z+i)^4 - 3$ conformal? Justify your response.*

2.1.(b) *Describe the geometric effects of $f(z)$ on the tangent vectors of the curves passing through the point $z = 1-2i$.*

2.2 *Let $\gamma : [0,3] \to \mathbb{C}$ be the contour given by*

$$\gamma(t) := \begin{cases} 2t, & \text{if } 0 \le t \le 1, \\ 4 - 2i + 2(-1+i)\,t, & \text{if } 1 \le t \le 2, \\ 2(3-t)\,i, & \text{if } 2 \le t \le 3. \end{cases}$$

(a) Sketch $\gamma(t)$ in $\mathbb{C}$.
(b) Evaluate $\int_\gamma \cos(z)dz$.

S.2.1.(a) We know from class that if a function f is holomorphic in a domain (i.e. open and connected set) U then it is conformal at $z_0 \in U$ if and only if $f'(z_0) \neq 0$. Our function f is an entire function so we only need to check where the derivative is zero.

$$f'(z) = 4(z+i)^3$$

which implies that $f'(z) \neq 0$ if and only if $z \neq -i$. Consequently, we conclude that f is conformal on $\mathbb{C} \setminus \{-i\}$.

2.1.(b) Given a function f on a domain U such that $f'(z_0) \neq 0$, we have that the tangent vectors of the image of a given curve passing by z_0 is the multiplication of the original tangent vector by $f'(z_0)$ which acts as a stretch by $|f'(z_0)|$ and rotation by $\operatorname{Arg}(f'(z_0))$. In our case $z_0 = 1-2i$ gives

$$f'(1-2i) = 4(1-i)^3 = 4\left(\sqrt{2}e^{-\frac{i\pi}{4}}\right)^3 = 2^{\frac{7}{2}}e^{-\frac{3i\pi}{4}}.$$

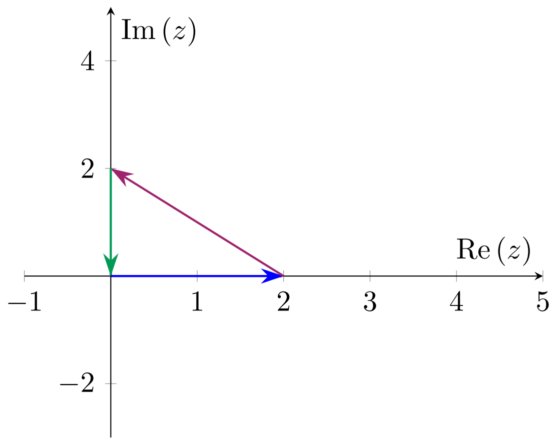
We conclude that the geometric effects of $f(z)$ on the tangent vectors of the curves passing through the point $z = 1-2i$ is a stretch by $2^{\frac{7}{2}}$ and rotation by $\frac{3\pi}{4}$ clockwise.

2.2.(a) The curve is composed of three straight lines which intersect at the appropriate points:

$$\gamma_1 : \quad x(t) = 2t, \quad y(t) = 0, \quad 0 \le t \le 1,$$

$$\gamma_2 : \quad x(t) = 4 - 2t, \quad y(t) = -2 + 2t, \quad 1 \le t \le 2,$$

$$\gamma_3 : \quad x(t) = 0, \quad y(t) = 2(3-t), \quad 2 \le t \le 3.$$



2.2.(b) We know that $\cos(z)$ is entire and as such is holomorphic in a domain that contains the closed contour γ ($\mathbb{C}$). By the Complex Fundamental Theorem of Calculus we conclude that

$$\int_\gamma \cos(z)dz = 0.$$

Q.3 *[Q5 from the May 2023 exam]*

3.1 Prove that for each $a \in \mathbb{R}$, $a > 0$, the series

$$\sum_{n=1}^{\infty} n^{-z}$$

converges uniformly on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1 + a\}$ where n^{-z} is defined using the principal logarithm [You may use without proof that $\sum_{n=1}^{\infty} n^{-b}$, $b \in \mathbb{R}$, $b > 1$, converges.]

3.2 Does the series $\sum_{n=1}^{\infty} n^{-z}$ defines a continuous function on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1\}$? Justify your response.

3.3 Does the series $\sum_{n=1}^{\infty} n^{-z}$ converge uniformly on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \geq 1\}$? Justify your response.

S.3 3.1 We will aim to use one of Weierstrass M-tests – standard or its local variant. Denote by $f_n(z) = n^{-z}$. We notice that for any $z \in \mathbb{C}$

$$|f_n(z)| = \left|e^{-z \operatorname{Log} n}\right| = \left|e^{-\mathrm{Re}(z)\log n} e^{-i\mathrm{Im}(z)\log n}\right| = e^{-\mathrm{Re}(z)\log n} = n^{-\mathrm{Re}(z)}.$$

On $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1 + a\}$ we have that

$$|f_n(z)| \leq n^{-1+a} = M_n.$$

Using the given hint we have that $\sum_{n=1}^{\infty} M_n < \infty$ and consequently, using Weierstrass' M-test, we conclude that $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1 + a\}$.

3.2 The idea is similar to the previous result, though we see that we can't avoid having $a > 0$ in using Weierstrass' M-test. However, we know that $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1 + a\}$ *for any* $a > 0$. Given $w \in \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1\}$ we can find $a_w > 0$ such that $w \in \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1 + a_w\}$, for example $a_w = \frac{1+\mathrm{Re}(w)}{2}$. In other words, for any $w \in \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1\}$ there exists an open set $U_w = \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1 + a_w\}$ that contains w and on which the series converges uniformly. This means, by definition, that $\sum_{n=1}^{\infty} f_n(z)$ converges locally uniformly on $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 1\}$. As $f_n(z)$ are continuous in the domain for any $n \in \mathbb{N}$ we conclude from a theorem from class that the resulting function is also continuous.

3.3 We notice that when $z = 1$ the series is nothing but the harmonic series $\sum_{n=1}^{\infty} n^{-1}$ which doesn't converge. Consequently the series can't converge uniformly in $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \geq 1\}$ as it doesn't even converge pointwise there.

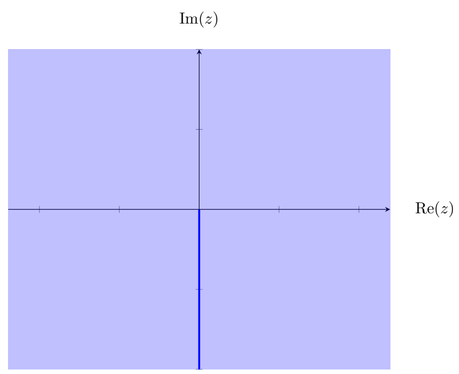
Q.4 *[Q6 from the May 2023 exam] Consider the set $U = \mathbb{C} \setminus \{iy \,:\, y \in \mathbb{R},\ y \leq 0\}$.*

4.1 Sketch the set U in $\mathbb{C}$.

4.2 Is U an open set? Justify your response.

4.3 Find a biholomorphic map from U to the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ and justify why this map is biholomorphic.

S.4 4.1



4.2 The set U is open. We can show it by definition, i.e. by showing that for any $z \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(z) \subset U$, or by showing that U^c is closed.
By definition: Given $z \in U$ we have that $\mathrm{Re}\,(z) \neq 0$ or $z = iy$ with $y > 0$. In the former case we choose $\epsilon = \frac{|\mathrm{Re}(z)|}{2}$ and find that for any $w \in B_\epsilon(z)$, $\mathrm{Re}(w) \neq 0$. Indeed, we have that

$$\mathrm{Re}\,(w) = \mathrm{Re} z + \mathrm{Re}\,(w - z)\,.$$

If $\mathrm{Re}(w) = 0$ then

$$\mathrm{Re}(z) = \mathrm{Re}\,(z - w)$$

which implies that

$$|\mathrm{Re}(z)| = |\mathrm{Re}\,(z - w)| \leq |z - w| < \frac{|\mathrm{Re}(z)|}{2},$$

which is impossible. Consequently, $B_\epsilon(z) \subset U$.
Consider now the case where $z = iy$ with $y > 0$. Then we claim that $B_{\frac{y}{2}}(z) \subset U$. Indeed, for any $w \in B_{\frac{y}{2}}(z)$

$$\mathrm{Im}(w) = \mathrm{Im}(z) + \mathrm{Im}(w - z) = y + \mathrm{Im}(w - z).$$

Since $|\mathrm{Im}(w - z)| \leq |w - z| < \frac{y}{2}$ we have that

$$\mathrm{Im}(w) > y - \frac{y}{2} = \frac{y}{2} > 0,$$

which shows that $w \in U$. As w was arbitrary we conclude that $B_{\frac{y}{2}}(z) \subset U$ and with it the openness of U.
By using U^c: By definition we find that

$$U^c = \mathbb{C} \setminus U = \{iy \;:\; y \in \mathbb{R},\; y \leq 0\}\,.$$

To show that U^c is closed we will show that if $\{z_n\}_{n\in\mathbb{N}} \subset U^c$ converges to a point $z \in \mathbb{C}$, then $z \in U^c$. Indeed, assuming that $\{z_n\}_{n\in\mathbb{N}}$ converges to z implies that

$$\mathrm{Im}\,(z_n) \underset{n\to\infty}{\longrightarrow} \mathrm{Im}(z).$$

Since $\{z_n\}_{n\in\mathbb{N}} \subset U$ we find that $\mathrm{Im}\,(z_n) \leq 0$ for all $n \in \mathbb{N}$. Consequently, $\mathrm{Im}(z) \leq 0$ as the limit of non-positive sequence. This implies that $z \in U^c$ and concludes the proof.

4.3 When considering maps to unit spheres powers and Möbius transformations come to mind. In our case we see that by rotating the domain by $\frac{\pi}{2}$ clockwise we get the domain

$$\{z \in \mathbb{C} \;:\; \mathrm{Re}(z) \leq 0\}$$

which lends itself well to powers that use the principal logarithm. This will allow us to use the principal squared root, $\sqrt{z}$, which will take the above domain to the right half plane. At this point we can rotate the domain by $\frac{\pi}{2}$ anti-clockwise we get the upper half plane and use the Cayley map which we know take the upper half plane to $\mathbb{D}$.
More formally: Consider the maps

$$f_1 : \mathbb{C} \to \mathbb{C}, \qquad f_1(z) = e^{-\frac{i\pi}{2}} z$$

$$f_2 : \mathbb{C} \setminus \{z \in \mathbb{C} \;:\; \mathrm{Re}(z) \leq 0\} \to \mathbb{H}_R, \qquad f_2(z) = \sqrt{z},$$

where the principal branch of the logarithm was chosen,

$$f_3 : \mathbb{H}_R \to \mathbb{H}, \qquad f_3(z) = e^{\frac{i\pi}{2}} z,$$

$$f_4 : \mathbb{H} \to \mathbb{D}, \qquad f_4(z) = \frac{z - i}{z + i}.$$

Each of these maps is holomorphic with a holomorphic inverse in the appropriate domain

$$f_1^{-1}(z) = f_3(z), \;\; f_2^{-1}(z) = z^2, \;\; f_3^{-1}(z) = f_1(z), \;\; f_4^{-1}(z) = i\frac{z + 1}{1 - z},$$

showing that each map is biholomorphic in the appropriate domain. The desired map would be

$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z).$$