Home Assignment 1

Exercise 1. Prove the following statement: Let $(\mathcal{X}, d)$ be a metric space where $\mathcal{X}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then the metric d is induced by a norm if and only if

(i) $d(x,y) = d\left(x+z, y+z\right)$ for any $x, y, z \in \mathcal{X}$.
(ii) $d\left(\alpha x, \alpha y\right) = |\alpha|\, d(x,y)$ for any $x, y \in \mathcal{X}$ and scalar α.

In that case the norm which induces the metric is given by

$$\|x\| = d(x, 0).$$

Exercise 2. Prove the following statement: Let (X, d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be Cauchy. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded

Exercise 3. Prove the following statement: Let (X, d) be a metric space and let A be a set in A. Show that A is dense in X if and only if for any $x \in X$ there exists a sequence of elements from A, $\{x_n\}_{n\in\mathbb{N}}$, that converges to x.

Exercise 4. Prove the following statement: Let (X, d) be a metric space. Show that X is separable if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that every $x \in X$ is a limit of a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to x.
Hint: Be careful here! Remember that a subsequence of a sequence must has increasing indexes.

Exercise 5. Show that if A is dense in a metric space (X, d) and if $A \subset B$, then B is dense in X.

Exercise 6. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space and let $\mathcal{M}$ be a subspace of $\mathcal{X}$. Show that $\overline{\mathcal{M}}$ is a subspace. Furthermore, show that for any set $M \subseteq \mathcal{X}$ we have that $\overline{\text{span} M}$ is the smallest closed subspace containing M, i.e. $\overline{\text{span} M}$ is a closed subspace and if $\mathcal{N}$ is a closed subspace of $\mathcal{X}$ that contains M then

$$\overline{\text{span} M} \subseteq \mathcal{N}.$$

Exercise 7. Show that the norm in $\left(\mathbb{F}^n, \|\cdot\|_p\right)$ is not induced by an inner product when $p \neq 2$.

Exercise 8. Show that the norm in $\left(L^p(E), \|\cdot\|_p\right)$ is not induced by an inner product when $p \neq 2$.

Exercise 9. Consider the space $\ell_p(\mathbb{N})$ defined in class.

(i) Show that $\ell_p(\mathbb{N})$ is closed under the addition and scalar multiplication, i.e. if $\boldsymbol{a}, \boldsymbol{b} \subset \ell_p(\mathbb{N})$ and α is a scalar then $\boldsymbol{a} + \boldsymbol{b}$ and $\alpha \boldsymbol{a}$ are in

$\ell_p(\mathbb{N})$.
Hint: You may use without proof the inequality

$$\left(|x|+|y|\right)^p \le 2^{p-1}\left(|x|^p+|y|^p\right)$$

which holds for any $x, y \in \mathbb{C}$ and $p \in [1,\infty)$.

(ii) Show that $\|\cdot\|_p$ is a norm on $\ell_p(\mathbb{N})$.
Hint: You may without proof use the following discrete Minkowski inequality:

$$\left(\sum_{n\in\mathbb{N}} |a_n+b_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n\in\mathbb{N}} |a_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n\in\mathbb{N}} |b_n|^p\right)^{\frac{1}{p}}$$

where $1 \le p < \infty$ and q is its Hölder conjugate.

Let $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}} \subset \ell_p(\mathbb{N})$ be a Cauchy sequence in $\ell_p(\mathbb{N})$ [1].

(iii) Show that for any $j \in \mathbb{N}$ we have that $\{a_{n,j}\}_{n\in\mathbb{N}}$, where $a_{n,j} = (\boldsymbol{a}_n)_j$, is a Cauchy sequence in $\mathbb{F}$ and conclude that it converges to some element $a_j \in \mathbb{F}$.
Hint: Show that for any $\boldsymbol{a}, \boldsymbol{b} \in \ell_p$, with $1 \le p \le \infty$, we have that for any $j \in \mathbb{N}$

$$|a_j - b_j| \le \|\boldsymbol{a}-\boldsymbol{b}\|.$$

(iv) Denoting by $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{N}}$. with a_j found in the previous sub-question show that $\boldsymbol{a} \in \ell_p(\mathbb{N})$. Moreover, show that for any $N \in \mathbb{N}$

$$\sum_{j=1}^{N} |a_j - a_{n,j}|^p \le \liminf_{m\to\infty} \|\boldsymbol{a}_m - \boldsymbol{a}_n\|_p^p$$

when $1 \le p < \infty$ and

$$\sup_{j\le N} |a_j - a_{n,j}| \le \liminf_{m\in\mathbb{N}} \|\boldsymbol{a}_m - \boldsymbol{a}_n\|_\infty.$$

Conclude that $\left(\ell_p(\mathbb{N}), \|\cdot\|_p\right)$ is Banach space.

Exercise 10. Consider the space $(C[a,b], \|\cdot\|_\infty)$ defined in class.

(i) Show that $C[a,b]$ is a vector space under pointwise addition and pointwise scalar multiplication.

(ii) Show that the function $\|\cdot\|_\infty : C[a,b] \to \mathbb{R}_+$ defined by

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

is a norm on $C[a,b]$.

(iii) Show that $C[a,b]$ is complete under the norm induced by $\|\cdot\|_\infty$, i.e. $C[a,b]$ is a Banach space.

[1] We have a sequence of sequences here, be mindful of the index!

(iv) Show that the norm of the space $C[a,b]$ is not induced by an inner product.