Solution to Home Assignment 1

Solution to Question 1. We start by assuming that the metric d is induced by a norm. In that case for any $x, y, z \in \mathcal{X}$

$$d\left(x,y\right) = \left\|x-y\right\| = \left\|(x+z)-\left(y+z\right)\right\| = d\left(x+z, y+z\right),$$

and any $x, y \in \mathcal{X}$ and scalar α

$$d\left(\alpha x, \alpha y\right) = \left\|\alpha x - \alpha y\right\| = |\alpha| \left\|x-y\right\| = |\alpha|\, d(x,y).$$

Conversely, we assume that the two properties are valid and define

$$\|x\| = d\left(x, 0\right).$$

We find that:

- $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $d\left(x, 0\right) = 0$ which holds if and only if $x = 0$.
- For any $x \in \mathcal{X}$ and a scalar α we have that
$$\|\alpha x\| = d\left(\alpha x, 0\right) = d\left(\alpha x, \alpha 0\right) = |\alpha|\, d\left(x, 0\right) = |\alpha| \, \|x\|.$$
- For any $x, y, z \in \mathcal{X}$ we have that
$$\left\|x+y\right\| = d\left(x+y, 0\right) = d\left(x+y-y, 0-y\right) = d\left(x, -y\right)$$
$$\leq d\left(x,0\right) + d\left(-y, 0\right) = \|x\| + \left\|-y\right\| = \|x\| + |-1| \left\|y\right\| = \|x\| + \left\|y\right\|.$$

As all three criteria for being a norm are satisfied, we conclude that $\|\cdot\|$ is indeed a norm.

Solution to Question 2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. There exists $n_0 \in \mathbb{N}$ such that for any $n, m \geq n_0$ we have that

$$d\left(x_n, x_m\right) < 1.$$

Consequently, for any $x_0 \in X$ and any $n \geq n_0$

$$d\left(x_n, x_0\right) \leq d\left(x_{n_0}, x_0\right) + d\left(x_n, x_{n_0}\right) < d\left(x_{n_0}, x_0\right) + 1.$$

We conclude that

$$d\left(x_n, x_0\right) \leq \max\left\{d\left(x_1, x_0\right), \dots, d\left(x_{n_0-1}, x_0\right), d\left(x_{n_0}, x_0\right) + 1\right\}.$$

As the right hand side is bounded uniformly in $n \in \mathbb{N}$ we conclude the desired boundedness.

Solution to Question 3. We start by assuming that A is dense in X. This means that $\overline{A} = X$. Given $x \in X$ we have that $x \in \overline{A}$ and as such we can find a sequence of elements from A, $\{x_n\}_{n\in\mathbb{N}}$, that converges to x.
Conversely, assuming that for any $x \in X$ we can a sequence of elements from A, $\{x_n\}_{n\in\mathbb{N}}$, that converges to x we see that any given $x \in X$ is a limit

point of A, or $x \in \overline{A}$. Since x was arbitrary we find that $\overline{A} = X$ which shows the density.

Solution to Question 4. Denote by $A = \{x_n\}_{n\in\mathbb{N}}$. If every $x \in X$ is a limit of a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ then every $x \in X$ is a limit of a sequence of elements from A. Consequently, according to the previous question, A is dense in X. Since A is countable we conclude that X is separable.
The converse is a bit more delicate. Assume that A is a countable dense set. Since A is countable we can write is as a sequence $A = \{x_n\}_{n\in\mathbb{N}}$. According to our previous question we know that every $x \in X$ is a limit point of A - but this doesn't immediately mean that the sequence we extract from A is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ in its current ordering! Let $\left\{a_j\right\}_{j\in\mathbb{N}} \subset A$ be the sequence that converges to x. By definition, for any $j \in \mathbb{N}$ we can find $n_j \in \mathbb{N}$ such that $a_j = x_{n_j}$. What we need to do is extract a subsequence of $\left\{a_j\right\}_{j\in\mathbb{N}}$, $\left\{a_{j_k}\right\}_{k\in\mathbb{N}}$, such that $\left\{n_{j_k}\right\}_{k\in\mathbb{N}}$ is increasing. This way $\left\{x_{n_{j_k}}\right\}_{k\in\mathbb{N}}$ will converge to x as a subsequence of $\left\{a_j\right\}_{j\in\mathbb{N}}$ *and* will be a subsequence of the original $\{x_n\}_{n\in\mathbb{N}}$.
We start by choosing $n_{j_1} = n_1$. As $\left\{n_j\right\}_{j\in\mathbb{N}}$ must go to infinity as j goes to infinity we can find $j_2 \in \mathbb{N}$ such that $n_{j_2} > n_1$. We continue inductively: n_{j_k} is chosen so that $n_{j_k} > n_{j_{k-1}}$ and since $\left\{n_j\right\}_{j\in\mathbb{N}}$ must go to infinity as j goes to infinity we can find $j_{k+1} \in \mathbb{N}$ such that $n_{j_{k+1}} > n_{j_k}$. This concludes the proof.

Solution to Question 5. Assume that A is dense in X and let B be such that $A \subset X$. For any $x \in X$ we can find a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ that converges to x. Since $A \subset B$ we conclude that for any $x \in X$ we can find a sequence $\{a_n\}_{n\in\mathbb{N}} \subset B$ that converges to x. This implies that B is dense, which is the desired result.

Solution to Question 6. To show that $\overline{\mathscr{M}}$ is a subspace we need to show that it is not empty and closed under addition and scalar multiplication. Since $\mathscr{M} \subseteq \overline{\mathscr{M}}$ and $\mathscr{M}$ is not empty, we find that $\overline{\mathscr{M}}$ is not empty.
Next, let $x, y \in \overline{\mathscr{M}}$. We can find sequences of elements in $\mathscr{M}$, $\{x_n\}_{n\in\mathbb{N}}$ and $\left\{y_n\right\}_{n\in\mathbb{N}}$, that converge to x and y respectively. Since $\mathscr{M}$ is a subspace we have that the sequence $\left\{x_n + y_n\right\}_{n\in\mathbb{N}}$ is in $\mathscr{M}$ and since it converges to $x + y$ we conclude that $x + y \in \overline{\mathscr{M}}$.
Similarly, for any $x \in \overline{\mathscr{M}}$ and any scalar α we find a sequence of elements in $\mathscr{M}$, $\{x_n\}_{n\in\mathbb{N}}$, that converges to x. As $\mathscr{M}$ is a subspace the sequence $\{\alpha x_n\}_{n\in\mathbb{N}}$ is in $\mathscr{M}$ and since it converges to αx we conclude that $\alpha x \in \overline{\mathscr{M}}$. The first part of the question is thus proved.
To show the second part we notice that the fact that $\overline{\text{span} M}$ is a closed subspace follows immediately from the above proof. We are only left to

show that it is the smallest closed subspace that contains M. Indeed, let $\mathcal{N}$ be a closed subspace that contains M. By definition this means that $\text{span}M \subseteq \mathcal{N}$ (since $\text{span}M$ is the smallest subspace that contains M). Since $\mathcal{N}$ is also a closed set we find that

$$\overline{\text{span}M} \subseteq \mathcal{N}$$

which concludes the proof.

Solution to Question 7. In order to show that a norm is not induced from an inner product we will show that the parallelogram identity is not satisfied for some vectors. Consider the standard basis $\left(\boldsymbol{e}_j\right)_{j=1,\dots,n} \subset \mathbb{F}^n$ where $\boldsymbol{e}_j$ is the vector whose entries are zero besides the entry in the $j-$th position, which is 1. We have that if $k \not j$ then

$$\left\|\boldsymbol{e}_j - \boldsymbol{e_k}\right\|_p = \left\|\boldsymbol{e}_j + \boldsymbol{e_k}\right\|_p = (1+1)^{\frac{1}{p}}.$$

Since $\left\|\boldsymbol{e}_j\right\|_p = \|\boldsymbol{e_k}\|_p = 1$ we see that the parallelogram identity i satisfied if and only if

$$2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2 + 2.$$

This holds if and only if $p = 2$.

Solution to Question 8. Similar to the question before, in order to show that a norm is not induced from an inner product we will show that the parallelogram identity is not satisfied for some vectors. The idea is the same - find two vectors with "disjoint support". Indeed, let A and B be measurable sets in E with a the same finite measure μ. Define

$$f = \chi_A, \qquad g = \chi_B.$$

We have that

$$\|f-g\|_p^p = \int_E \left|\chi_A(x) - \chi_B(x)\right|^p dx = \int_E \left(\chi_A(x)^p + \chi_B(x)^p\right) dx$$

$$= \int_E \left(\chi_A(x) + \chi_B(x)\right) dx = 2\mu.$$

Similarly $\|f+g\|_p^p = 2\mu$. Since $\|f\|_p = \|g\|_p = \mu^{\frac{1}{p}}$ we see that the parallelogram identity i satisfied if and only if

$$(2\mu)^{\frac{2}{p}} + (2\mu)^{\frac{2}{p}} = 2\mu^{\frac{2}{p}} + 2\mu^{\frac{2}{p}}.$$

This holds if and only if $p = 2$ (it is, in fact, the same identity as in the previous question).

Solution to Question 9. (i) We start with $1 \le p < \infty$. For any $N \in \mathbb{N}$ and any $\boldsymbol{a}, \boldsymbol{b} \in \ell_p(\mathbb{N})$ we have that

$$\sum_{n=1}^{N} |a_n + b_n|^p \le \sum_{n=1}^{N} (|a_n| + |b_n|)^p \le 2^{p-1}\left(\sum_{n=1}^{N} |a_n|^p + \sum_{n=1}^{N} |b_n|^p\right)$$
$$\le 2^{p-1}\left(\|\boldsymbol{a}\|^p + \|\boldsymbol{b}\|^p\right).$$

As this holds for every $N \in \mathbb{N}$ and the right hand side is independent of N, taking N to infinity shows that

$$\sum_{n\in\mathbb{N}} |a_n + b_n|^p \le 2^{p-1}\left(\sum_{n\in\mathbb{N}} |a_n|^p + \sum_{n\in\mathbb{N}} |b_n|^p\right) < \infty$$

which proves that $\boldsymbol{a} + \boldsymbol{b} \in \ell_p(\mathbb{N})$. Similarly, for any $\boldsymbol{a} \in \ell_p(\mathbb{N})$ and a scalar α we have that

$$\sum_{n=1}^{N} |\alpha a_n|^p = |\alpha|^p \sum_{n=1}^{N} |a_n|^p \underset{N\to\infty}{\longrightarrow} |\alpha|^p \sum_{n\in\mathbb{N}} |a_n|^p = |\alpha|^p \|\boldsymbol{a}\|_p^p < \infty$$

which implies that $\alpha\boldsymbol{a} \in \ell_p(\mathbb{N})$ and $\|\alpha\boldsymbol{a}\| = |\alpha| \|\boldsymbol{a}\|_p$.
The case $p = \infty$ is more straightforward to show since

$$\|\boldsymbol{a} + \boldsymbol{b}\|_\infty = \sup_{n\in\mathbb{N}} |a_n + b_n| \le \sup_{n\in\mathbb{N}} (|a_n| + |b_n|) \le \sup_{n\in\mathbb{N}} |a_n| + \sup_{n\in\mathbb{N}} |b_n|$$
$$= \|\boldsymbol{a}\|_\infty + \|\boldsymbol{b}\|_\infty < \infty$$

and for any scalar α

$$\|\alpha\boldsymbol{a}\|_\infty = \sup_{n\in\mathbb{N}} |\alpha a_n| = |\alpha| \sup_{n\in\mathbb{N}} |a_n| = |\alpha| \|\boldsymbol{a}\|_\infty < \infty.$$

(ii) To show that $\|\cdot\|_p$ is a norm we notice that in the previous sub-question we have shown the scaling property for the proposed norm, as well as the triangle inequality for the case where $p = \infty$. The triangle inequality for the case where $1 \le p < \infty$ is nothing more than the discrete Minkowski's inequality. Consequently, in order to show that $\|\cdot\|_p$ is indeed a norm we only need to show that it has the positivity property.
We start with the case $1 \le p < \infty$: By definition $\|\boldsymbol{a}\|_p \ge 0$.

$$\|\boldsymbol{a}\|_p = 0 \Leftrightarrow \sum_{n\in\mathbb{N}} |a_n|^p = 0 \underbrace{\Leftrightarrow}_{\substack{\text{non-negative}\\ \text{series}}} |a_n|^p = 0 \ \forall n \in \mathbb{N}$$
$$\Leftrightarrow a_n = 0 \ \forall n \in \mathbb{N} \Leftrightarrow \boldsymbol{a} = \boldsymbol{0}.$$

Similarly, for $p = \infty$: By definition $\|\boldsymbol{a}\|_\infty \ge 0$.

$$\|\boldsymbol{a}\|_\infty = 0 \Leftrightarrow \sup_{n\in\mathbb{N}} |a_n| = 0 \underbrace{\Leftrightarrow}_{\substack{\text{non-negative}\\ \text{sequence}}} |a_n| = 0 \ \forall n \in \mathbb{N}$$

$$\Leftrightarrow\ a_n = 0\ \forall n\in\mathbb{N}\ \Leftrightarrow\ \boldsymbol{a} = \boldsymbol{0}.$$

We thus conclude that $\|\cdot\|_p$ is indeed a norm on $\ell_p(\mathbb{N})$ for any $1 \le p \le \infty$.

(iii) We start by noticing that for any $\boldsymbol{a},\boldsymbol{b}\in\ell_p(\mathbb{N})$ we have that for any $j\in\mathbb{N}$

$$\left|a_j - b_j\right| \le \left(\sum_{j\in\mathbb{N}}\left|a_j-b_j\right|^p\right)^{\frac{1}{p}} = \|\boldsymbol{a}-\boldsymbol{b}\|_p$$

when $1\le p<\infty$ and

$$\left|a_j - b_j\right| \le \sup_{j\in\mathbb{N}}\left|a_j-b_j\right| = \|\boldsymbol{a}-\boldsymbol{b}\|_\infty$$

when $p=\infty$. In other words, for any $1\le p\le\infty$ and any $j\in\mathbb{N}$ we have that

$$\left|a_j - b_j\right| \le \|\boldsymbol{a}-\boldsymbol{b}\|_p.$$

Since the above holds for any $j\in\mathbb{N}$ we find that

$$\sup_{j\in\mathbb{N}}\left|a_j - b_j\right| \le \|\boldsymbol{a}-\boldsymbol{b}\|_p.$$

Given a Cauchy sequence in $\ell_p(\mathbb{N})$, $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ we have that for any $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that if $n,m\ge n_0$

$$\|\boldsymbol{a}_n-\boldsymbol{a}_m\|_p<\varepsilon.$$

Consequently, for any $n,m\ge n_0$

$$\sup_{j\in\mathbb{N}}\left|a_{n,j}-a_{m,j}\right| \le \|\boldsymbol{a}_n-\boldsymbol{a}_m\|_p<\varepsilon,$$

which shows that $\left\{a_{n,j}\right\}_{n\in\mathbb{N}}$ is Cauchy for any $j\in\mathbb{N}$ (in fact it is Cauchy *uniformly in* j!). Since this sequence is Cauchy in a complete space ($\mathbb{F}$) we know that there exists an element $a_j\in\mathbb{F}$ such that

$$a_{n,j}\underset{n\to\infty}{\longrightarrow}a_j.$$

(iv) We need to divide our consideration to two cases: $1\le p<\infty$ and $p=\infty$. When $1\le p<\infty$ we have that for any $N\in\mathbb{N}$

$$\sum_{j=1}^N\left|a_j\right|^p = \lim_{n\to\infty}\sum_{j=1}^N\left|a_{n,j}\right|^p = \liminf_{n\to\infty}\sum_{j=1}^N\left|a_{n,j}\right|^p \le \liminf_{n\to\infty}\|\boldsymbol{a}_n\|_p^p.$$

Since $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ is Cauchy in $\ell_p(\mathbb{N})$ it must be bounded, i.e. $\sup_{n\in\mathbb{N}}\|\boldsymbol{a}_n\|_p<\infty$ and the above implies that

$$\sum_{j=1}^N\left|a_j\right|^p \le \sup_{n\in\mathbb{N}}\|\boldsymbol{a}_n\|_p^p<\infty$$

for any $N \in \mathbb{N}$. As the right hand side is independent of N we can take it to infinity and get that

$$\|\boldsymbol{a}\|_p = \left(\sum_{j\in\mathbb{N}} \left|a_j\right|^p\right)^{\frac{1}{p}} \leq \sup_{n\in\mathbb{N}} \|\boldsymbol{a}_n\|_p < \infty$$

showing that $\boldsymbol{a}$ is in $\ell_p(\mathbb{N})$.
The case $p = \infty$ is similar but more straightforward: For any $j \in \mathbb{N}$

$$\left|a_j\right| = \lim_{n\to\infty} \left|a_{n,j}\right| = \liminf_{n\to\infty} \left|a_{n,j}\right| \leq \liminf_{n\to\infty} \|\boldsymbol{a}_n\|_\infty .$$

Consequently, as the right hand side is independent of j,

$$\|\boldsymbol{a}\|_\infty = \sup_{j\in\mathbb{N}} \left|a_j\right| \leq \liminf_{n\to\infty} \|\boldsymbol{a}_n\|_\infty \leq \sup_{n\to\infty} \|\boldsymbol{a}_n\|_\infty < \infty$$

which shows that $\boldsymbol{a} \in \ell_\infty(\mathbb{N})$.
Next we turn our attention to the requested inequality. Let $N \in \mathbb{N}$ be given and consider $p \in [1,\infty)$. Similarly to the proof above we find that

$$\sum_{j=1}^{N} \left|a_j - a_{n,j}\right|^p = \lim_{m\to\infty} \sum_{j=1}^{N} \left|a_{m,j} - a_{n,j}\right|^p$$

$$= \liminf_{m\to\infty} \sum_{j=1}^{N} \left|a_{m,j} - a_{n,j}\right|^p \leq \liminf_{n\to\infty} \|\boldsymbol{a}_m - \boldsymbol{a}_n\|_p^p .$$

When $p = \infty$ we have that

$$\sup_{j\leq N} \left|a_j - a_{n,j}\right| = \sup_{j\leq N} \lim_{m\to\infty} \left|a_{m,j} - a_{n,j}\right| = \sup_{j\leq N} \liminf_{m\to\infty} \left|a_{m,j} - a_{n,j}\right|$$

$$\leq \sup_{j\leq N} \liminf_{n\to\infty} \|\boldsymbol{a}_m - \boldsymbol{a}_n\|_\infty = \liminf_{n\to\infty} \|\boldsymbol{a}_m - \boldsymbol{a}_n\|_\infty .$$

We claim that these inequalities imply the convergence of $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ to $\boldsymbol{a}$. Indeed, given $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for any $n, m \geq n_0$ we have that

$$\|\boldsymbol{a}_m - \boldsymbol{a}_m\|_p < \varepsilon.$$

For any $n \geq n_0$ we have that

$$\sum_{j=1}^{N} \left|a_j - a_{n,j}\right|^p < \varepsilon^p$$

when $1 \leq p < \infty$ and

$$\sup_{j\leq N} \left|a_j - a_{n,j}\right| < \varepsilon$$

when $p = \infty$. As the right hand side in both cases is independent of N we conclude that for all $n \geq n_0$

$$\|\boldsymbol{a} - \boldsymbol{a}_n\|_p < \varepsilon,$$

which shows the convergence. As we have shown that any Cauchy sequence in $\ell_p(\mathbb{N})$ has a limit in $\ell_p(\mathbb{N})$ we conclude that $\ell_p(\mathbb{N})$ is indeed a Banach space.

Solution to Question 10. (i) This follows from arithmetic of continuous functions since the zero function is continuous, addition of continuous functions is a continuous function, and scalar multiplication of continuous functions is a continuous function.

(ii) We have that

- $\|f\|_\infty \geq 0$ by definition and $\|f\|_\infty = 0$ if and only if $\max_{x\in[a,b]} |f(x)| = 0$. Since $|f(x)|$ is non-negative we conclude that the above holds if and only if $f(x) = 0$ for all $x \in [a,b]$, or equivalently if $f \equiv 0$.
- For any scalar α we have that
$$\|\alpha f\|_\infty = \max_{x\in[a,b]} |\alpha f(x)| = |\alpha| \left(\max_{x\in[a,b]} |f(x)| \right) = |\alpha| \, \|f\|_\infty .$$
- For any $f, g \in C[a,b]$ we have that since for any $x \in [a,b]$
$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$
we have that
$$\|f+g\|_\infty = \max_{x\in[a,b]} |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty .$$

From the above we conclude that $\|\cdot\|_\infty$ is indeed a norm on $C[a,b]$.

(iii) Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(C[a,b], \|\cdot\|_\infty)$. Since for any $x \in [a,b]$
$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$
we conclude (just like in the case of $\ell_\infty(\mathbb{N})$) that $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$. Since $\mathbb{F}$ is complete we find that for any $x \in [a,b]$ there exists $f(x)$ in $\mathbb{F}$ such that $f_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$. Moreover,
$$\begin{aligned} |f(x) - f_n(x)| &= \lim_{m\to\infty} |f_m(x) - f_n(x)| = \liminf_{m\to\infty} |f_m(x) - f_n(x)| \\ &\leq \lim_{m\to\infty} \|f_m - f_n\|_\infty . \end{aligned}$$
This implies that
$$\|f - f_n\|_\infty \leq \lim_{m\to\infty} \|f_m - f_n\|_\infty$$
and consequently that $\{f_n\}_{n\in\mathbb{N}}$ converges in norm to f. We are only left with showing that f is in $C[a,b]$ to conclude that the space is complete and as such Banach. Since $\{f_n\}_{n\in\mathbb{N}}$ are all continuous and converge uniformly (that is what the $\|\cdot\|_\infty$ is) to f, a theorem from Analysis I guarantees us that f is also continuous, which is what we wanted to show.