Home Assignment 2

Exercise 1. Prove the following statement: Let $\mathscr{H}$ be an inner product space and let M be a subset of $\mathscr{H}$. Then

(i) $M^{\perp}$ is a subspace of $\mathscr{H}$.
(ii) $M^{\perp}$ is a closed set.
(iii) $M^{\perp} = \overline{M}^{\perp}$.
(iv) $M^{\perp} = \left(\text{span}M\right)^{\perp}$.
(v) $M^{\perp} = \overline{\text{span}M}^{\perp}$.
(vi) Let $M^{\perp\perp}$ be defined as $\left(M^{\perp}\right)^{\perp}$. Then, if $\mathscr{M}$ is a subspace of $\mathscr{H}$ we have that $\mathscr{M}^{\perp\perp} = \overline{\mathscr{M}}$.
Hint: You could use the projection theorem.

Exercise 2. Prove the following statement[1]: Every non-zero vector space $\mathscr{X}$ has a Hamel basis.
Hint: Define a partial order of inclusion on the set of all linearly independent sets in $\mathscr{X}$, i.e

$$A \leq B \qquad \text{if} \qquad A \subset B.$$

Show that the conditions of Zorn's lemma are satisfied in this case and conclude that a maximal element of this order must be a Hamel basis.

Exercise 3. Consider the set $\mathscr{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}} \subset \ell_p(\mathbb{N})$ defined by

$$\boldsymbol{e}_n = \left(0,\dots,0, \underbrace{1}_{n-\text{th poistion}}, 0, \dots\right).$$

Show that if $\boldsymbol{a} \in \text{span}\mathscr{B}$ then there exists $n_0 \in \mathbb{N}$ such that $a_n = 0$ for all $n > n_0$. Conclude that if $\boldsymbol{a} \in \ell_p(\mathbb{N})$ has no zero entries then $\boldsymbol{a} \notin \text{span}\mathscr{B}$.

Exercise 4. Show that the set $\mathscr{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ defined in the previous question is a Schauder basis for $\ell_p(\mathbb{N})$ when $1 \leq p < \infty$.
Hint: Recall that you've shown that for any $\boldsymbol{a}, \boldsymbol{b} \in \ell_p(\mathbb{N})$ and for any $n \in \mathbb{N}$ we have that

$$|a_n - b_n| \leq \|\boldsymbol{a} - \boldsymbol{b}\|_p,$$

and use that to show the uniqueness of the expansion with respect to $\mathscr{B}$.

Exercise 5. Let $\mathscr{X}$ be a Banach space with a Schuader basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$. Is the set $\mathscr{B}_1 = \left\{\frac{e_n}{\|e_n\|}\right\}_{n\in\mathbb{N}}$ also a Schauder basis for $\mathscr{X}$?

[1] This is a challenging exercise!

Exercise 6. We say that a Shcauder basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$ to a Banach space $\mathscr{X}$ is *unconditional* if whenever $\sum_{n\in\mathbb{N}} a_n e_n$ converges so does $\sum_{n\in\mathbb{N}} a_{\sigma(n)} e_{\sigma(n)}$ for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ (i.e. the order of summation doesn't impact the convergence of the series). Show that the standard basis for $\ell_p(\mathbb{N})$, with $1 \leq p < \infty$, $\mathscr{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$, defined in Exercise 4 is an unconditional basis.

Exercise 7. Prove the following statement: Let $\mathscr{X}$ be a normed space over $\mathbb{R}$ or $\mathbb{C}$. If M is a countable set and span$M$ is dense in $\mathscr{X}$ then $\mathscr{X}$ is separable.

Exercise 8. Show that $\ell_\infty(\mathbb{N})$ can't have a Schauder basis.

Exercise 9. Consider the Hilbert space $L^2[-\pi,\pi]$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

Use the fact that $\{e^{inx}\}_{n\in\mathbb{Z}}$ (notice $n \in \mathbb{Z}$ and not in $\mathbb{N}$!) is an orthonormal basis for $L^2[-\pi,\pi]$ (no proof is required) and show that

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hint: Consider the function $f(x) = x$.

Exercise 10. Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Show that there is no orthonormal basis for $\mathscr{H}$ that is a Hamel basis. Conclude that Hilbert spaces can't have a countable Hamel basis.

Exercise 11. Prove the following statement: Let $\mathscr{H}$ be an inner product space and let $\mathscr{B} = \{e_\alpha\}_{\alpha\in\mathscr{I}}$ be orthonormal. If $\mathscr{I}$ is uncountable, then for any $x \in \mathscr{H}$ we have that $\langle x, e_\alpha\rangle \neq 0$ for at most a countable subset of $\mathscr{B}$, $\{e_{\alpha_n}\}_{n\in\mathbb{N}}$.
Hint: For a given $x \in \mathscr{H}$ and $k \in \mathbb{N}$ consider the set

$$M_k(x) = \left\{ i \in \mathscr{I} \mid |\langle x, e_i\rangle| \geq \frac{1}{k} \right\}$$

and use Bessel's inequality to show that $M_k(x)$ must be finite.

Exercise 12. Let $\mathscr{M}$ be a closed subspace of a Hilbert space $\mathscr{H}$. Show that $\mathscr{M} = \mathscr{H}$ if and only if $\mathscr{M}^\perp = \{0\}$.

Exercise 13. Prove the following statement: Every non-trivial Hilbert space has an orthonormal basis.
Hint: Use Zorn's lemma on the collection of orthonormal sets and the inclusion partial order.