Solution to Home Assignment 2

Solution to Question 1. (i) We notice that 0 is perpendicular to any vector and as such $0 \in M^\perp$. Next we notice that if $x, y \in M^\perp$ then for any $m \in M$

$$\langle x+y, m\rangle = \langle x, m\rangle + \langle y, m\rangle = 0+0=0$$

which shows that $x+y \in M^\perp$. Lastly, if $x \in M^\perp$ and α is a scalar then

$$\langle \alpha x, m\rangle = \alpha \langle x, m\rangle = \alpha 0 = 0$$

from which we conclude that $\alpha x \in M^\perp$. As $M^\perp$ is not empty and closed under addition and scalar multiplication we conclude that it must be a subspace.

(ii) Let $\{x_n\}_{n\in\mathbb{N}} \subset M^\perp$ be a given sequence that converges to some $x \in \mathscr{H}$. Let $m \in M$ be given. Using the continuity of the inner product we find that

$$\langle x, m\rangle = \lim_{n\to\infty} \langle x_n, m\rangle \underset{x_n\in M^\perp}{=} \lim_{n\to\infty} 0 = 0.$$

As the above holds for any $m \in M$ we find that $x \in M^\perp$. Thus, all limits of sequences from $M^\perp$ are in $M^\perp$ which shows that it is a closed set.

(iii) One inclusion in the identity is immediate. Indeed, since $M \subset \overline{M}$ we have that if $x \perp y$ for all $y \in \overline{M}$ then $x \perp y$ for all $x \in M$, i.e. $\overline{M}^\perp \subset M^\perp$. Notice that we have in fact shown that if $A \subset B$ then $B^\perp \subset A^\perp$.
Let us consider the other inclusion. Let $x \in M^\perp$ and let $y \in \overline{M}$. We know that we can find a sequence of elements in M, $\{y_n\}_{n\in\mathbb{N}}$, such that $y_n \underset{n\to\infty}{\longrightarrow} y$. Using the continuity of the inner product we see that

$$\langle x, y\rangle = \lim_{n\to\infty} \langle x, y_n\rangle \underset{x\in M^\perp}{=} \lim_{n\to\infty} 0 = 0,$$

which shows that $x \perp y$ for all $y \in \overline{M}$. Thus $M^\perp \subset \overline{M}^\perp$. Combining this with the other inclusion proves the result.

(iv) As we saw in the previous sub-question proof, since $M \subset \operatorname{span} M$ we have that

$$(\operatorname{span} M)^\perp \subset M^\perp.$$

Conversely, assume that $x \in M^\perp$ and let $y \in \operatorname{span} M$. By definition, there exist $y_1, \dots, y_n \in M$ and scalars $\alpha_1, \dots, \alpha_n$ such that

$$y = \sum_{i=1}^n \alpha_i y_i.$$

As such

$$\langle x,y\rangle = \left\langle x, \sum_{i=1}^{n} \alpha_i y_i \right\rangle = \sum_{i=1}^{n} \overline{\alpha_i} \langle x, y_i\rangle \underset{x\in M^\perp}{=} \sum_{i=1}^{n} \overline{\alpha_i} \cdot 0 = 0.$$

Since y was arbitrary we conclude that $x \in (\mathrm{span} M)^\perp$ which shows that $M^\perp \subset (\mathrm{span} M)^\perp$.

(v) This follows immediately from previous sub-questions.

(vi) For any given set A, since

$$A^\perp = \{x \in \mathscr{H} \mid x \perp y, \ \forall y \in A\}$$

we see that for any $y \in A$ and $x \in A^\perp$ we have that $\langle x, y\rangle = 0$. This implies that $A \subset A^{\perp\perp}$. Since $A^{\perp\perp} = (A^\perp)^\perp$, a previous sub-question guarantees that $A^{\perp\perp}$ is closed and consequently the inclusion we've shown implies that $\overline{A} \subset A^{\perp\perp}$. Note that this inclusion is alwasy true and doesn't require the set to be a subspace.

We shall now focus on showing the converse inclusion when the set $\mathscr{M}$ is a subspace. Let $\widetilde{m} \in \mathscr{M}^{\perp\perp}$. Since $\overline{\mathscr{M}}$ is a closed subspace in $\mathscr{H}$ (as you saw in the previous assignment) we know that $P_{\overline{\mathscr{M}}}\widetilde{m} \in \overline{\mathscr{M}}$ exists. Moreover,

$$\widetilde{m} - P_{\overline{\mathscr{M}}}\widetilde{m} \in \overline{\mathscr{M}}^\perp.$$

On the other hand, since $\overline{\mathscr{M}} \subset \mathscr{M}^{\perp\perp}$, and $\mathscr{M}^{\perp\perp}$ is a subspace, we see that

$$\widetilde{m} - P_{\overline{\mathscr{M}}}\widetilde{m} \in \mathscr{M}^{\perp\perp}.$$

Using the fact that $\overline{\mathscr{M}}^\perp = \mathscr{M}^\perp$ we conclude that

$$\widetilde{m} - P_{\overline{\mathscr{M}}}\widetilde{m} \in \mathscr{M}^\perp \cap (\mathscr{M}^\perp)^\perp = \{0\}.$$

In other words, $\widetilde{m} = P_{\overline{\mathscr{M}}}\widetilde{m} \in \overline{M}$. As $\widetilde{m}$ was arbitrary we find that $\mathscr{M}^{\perp\perp} \subset \overline{M}$ and conclude the proof.

Solution to Question 2. Let $\mathcal{M}$ be the set of all linearly independent sets in $\mathcal{X}$. $\mathcal{M} \neq \emptyset$ since $\mathcal{X}$ is not the trivial vector space. We define a partial order of inclusion on $\mathcal{M}$:

$$A \le B \qquad \text{if} \qquad A \subset B,$$

We claim that any maximal element of this partial order must be a Hamel basis for $\mathcal{X}$. Indeed, if $\mathscr{B}$ is a maximal element of $\mathcal{M}$ and $\mathcal{X} \neq \mathrm{span}\mathscr{B}$ then there exists $x \in \mathcal{X} \setminus \mathrm{span}\mathscr{B}$. The set $\widetilde{\mathscr{B}} = \mathscr{B} \cup \{x\}$ is independent and satisfies $\mathscr{B} \le \widetilde{\mathscr{B}}$. However, as $\mathscr{B} \neq \widetilde{\mathscr{B}}$ and $\mathscr{B}$ is maximal, we have reached a contradiction.

To show that we have a maximal element we will invoke Zorn's lemma. Let $\mathcal{C}$ be a chain in $\mathcal{M}$ and define

$$U = \cup_{A\in\mathcal{C}} A.$$

We claim that U is independent. Indeed, if $x_1,\dots,x_n \in U$ then there exist $A_1,\dots,A_n \in \mathcal{C}$ such that $x_i \in A_i$ for $i = 1,\dots,n$. Since $A_1,\dots,A_n$ are in a chain, they have a maximum. Without loss of generality, this set is A_1, and as such for all $i = 1,\dots,n$ we have that $x_i \in A_i \subset A_1$. Since A_1 is independent we find that $\{x_1,\dots,x_n\}$ are independent. As, $x_1,\dots,x_n \in U$ were arbitrary we see that every finite collection of vectors in U are independent, or equivalently - U is independent. This implies that $U \in \mathcal{M}$. By the definition of the partial order of $\mathcal{M}$ we have that $A \le U$ for any $A \in \mathcal{C}$ and we conclude that every chain in $\mathcal{M}$ have an upper bound. Thus we can use Zorn's lemma and conclude the proof.

Solution to Question 3. Assume that $\boldsymbol{a} \in \text{span}\mathscr{B}$. Then, there exists $k \in \mathbb{N}$, $n_1,\dots,n_k \in \mathbb{N}$, and scalars $\alpha_{n_1},\dots,\alpha_{n_k}$ such that

$$\boldsymbol{a} = \sum_{i=1}^{k} \alpha_{n_i} \boldsymbol{e}_{n_i}.$$

Let $n_0 = \max\{n_1,\dots,n_k\}$. Since $\left(\boldsymbol{e}_{n_i}\right)_j = 0$ for any $j > n_0$ we see that

$$a_j = \left(\sum_{i=1}^{k} \alpha_{n_i} \boldsymbol{e}_{n_i}\right)_j = \sum_{i=1}^{k} \alpha_{n_i} \left(\boldsymbol{e}_{n_i}\right)_j = 0$$

for all $j > n_0$. Consequently, if $\boldsymbol{a}$ has no zero entries it can't be in $\text{span}\mathscr{B}$.

Solution to Question 4. We have seen in class that for any $\boldsymbol{a} = (a_1, a_2, \dots) \in \ell_p(\mathbb{N})$ we have that the partial sums sequence

$$S_N = \sum_{n=1}^{N} a_n \boldsymbol{e}_n$$

converges to $\boldsymbol{a}$. To show that $\mathscr{B}$ is a Schauder basis we only need to show the uniqueness of the coefficients. Indeed, assume that $\widetilde{S}_N = \sum_{n=1}^{N} \alpha_n(\boldsymbol{a})\, \boldsymbol{e}_n$ converges to $\boldsymbol{a}$ in $\ell_p(\mathbb{N})$. By definition of the $\ell_p(\mathbb{N})$ norm we have that for any $n \in \mathbb{N}$ and any $\boldsymbol{a}, \boldsymbol{b} \in \ell_p(\mathbb{N})$

$$|a_n - b_n| \le \left(\sum_{n\in\mathbb{N}} |a_n - b_n|^p\right)^{\frac{1}{p}} = \|\boldsymbol{a} - \boldsymbol{b}\|_p.$$

Consequently, for any $n \in \mathbb{N}$ and any $N \ge n$ we have that

$$|a_n - \alpha_n(\boldsymbol{a})| \le \left\|\boldsymbol{a} - \widetilde{S}_N\right\|_p.$$

Taking N to we find that

$$0 \leq |a_n - \alpha_n(\boldsymbol{a})| \leq \lim_{N\to\infty} \left\| \boldsymbol{a} - \widetilde{S}_N \right\|_p = 0.$$

Thus $\alpha_n(\boldsymbol{a}) = a_n$ which shows the uniqueness of the partial sums expansion.

Solution to Question 5. We start by noticing that since $\mathscr{B}$ is independent it can't contain the zero vector. This implies that $e_n \neq 0$ for all $n \in \mathbb{N}$ and consequently that $\|e_n\| \neq 0$ for all $n \in \mathbb{N}$. This shows that $\mathscr{B}_1$ is indeed well defined. Denoting by $x_n = \frac{e_n}{\|e\|_n}$ we see that $\mathscr{B}_1 = \{x_n\}_{n\in\mathbb{N}}$ is independent as x_n is just a scalar multiple of e_n, which are independent.
Moreover, as $\mathscr{B}$ is Schauder, for any $x \in \mathscr{X}$ we can find a unique sequence of scalars, $\{\alpha_n(x)\}_{n\in\mathbb{N}}$ such that the partial sum sequence $\{S_N(x)\}_{N\in\mathbb{N}}$ with

$$S_N(x) = \sum_{n=1}^{N} \alpha_n(x) e_n$$

converges to x. Defining $\beta_n(x) = \|e_n\| \alpha_n(x)$ we see that

$$\widetilde{S}_N(x) = \sum_{n=1}^{N} \beta_n(x) x_n = \sum_{n=1}^{N} (\|e_n\| \alpha_n(x)) \frac{e_n}{\|e_n\|} = S_N(x)$$

which converges to x. Thus, in order to show that $\mathscr{B}_1$ is a Schauder basis we only need to show the uniqueness of the coefficients. Indeed, assume that

$$T_N(x) = \sum_{n=1}^{N} \gamma_n(x) x_n$$

converges to x. Since

$$T_N(x) = \sum_{n=1}^{N} \left(\frac{\gamma_n(x)}{\|e_n\|} \right) e_n$$

the uniqueness of the coefficients in the expansion with respect to $\mathscr{B}$ imply that $\frac{\gamma_n(x)}{\|e_n\|} = \alpha_n(x)$. Consequently, $T_N(x) = \widetilde{S}_N(x)$ and the uniqueness of the coefficients with respect to $\mathscr{B}_1$ has been shown. We conclude that $\mathscr{B}_1$ is indeed a Schauder basis.

Solution to Question 6. Given $\boldsymbol{a} = (a_1, \ldots, a_n, \ldots) \in \ell_p(\mathbb{N})$ and $\sigma : \mathbb{N} \to \mathbb{N}$ we define

$$S_N = \sum_{n=1}^{N} a_{\sigma(n)} \boldsymbol{e}_{\sigma(n)}.$$

We have that

$$\|S_N - S_M\|_p^p = \left\| \sum_{n=\min(N,M)+1}^{\max(N,M)} a_{\sigma(n)} \boldsymbol{e}_{\sigma(n)} \right\|^p$$

$$= \sum_{n=\min(N,M)+1}^{\max(N,M)} \left|a_{\sigma(n)}\right|^p.$$

For a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $N \geq n_0$

$$\sum_{n=N}^{\infty} |a_n|^p < \varepsilon^p.$$

Since σ is a bijection, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have that $\sigma(n) \geq n_0$ and consequently, if $\min(N, M) \geq n_1$ we have that

$$\|S_N - S_M\|_p^p = \sum_{n=\min(N,M)+1}^{\max(N,M)} \left|a_{\sigma(n)}\right|^p < \varepsilon^p.$$

As $\varepsilon > 0$ was arbitrary $\{S_N\}_{N\in\mathbb{N}}$ is Cauchy and since $\ell_p(\mathbb{N})$ is a Banach space, the sequence converges, which is what we wanted to show.

Solution to Question 7. ince M is countable we can find a sequence $\{e_n\}_{n\in\mathbb{N}}$ such that $M = \{e_n\}_{n\in\mathbb{N}}$. Define the countable set

$$\mathscr{M}_n = \begin{cases} \left\{\sum_{i=1}^n q_i e_i \mid q_i \in \mathbb{Q}\right\}, & \mathbb{F} = \mathbb{R}, \\ \left\{\sum_{i=1}^n q_i e_i \mid q_i \in \mathbb{Q} + i\mathbb{Q}\right\}, & \mathbb{F} = \mathbb{C}, \end{cases}$$

and the set $\mathscr{M} = \cup_{n\in\mathbb{N}} \mathscr{M}_n$. $\mathscr{M}_n$ is countable for any $n \in \mathbb{N}$ and consequently $\mathscr{M}$ is also countable.
Next, we define

$$\mathscr{X}_n = \operatorname{span}\{e_1, \ldots, e_n\}$$

and find that since $\mathbb{Q}$ and $\mathbb{Q} + i\mathbb{Q}$ are dense in $\mathbb{R}$ and $\mathbb{C}$ respectively, and since $\mathscr{X}_n$ is spanned by finitely many vectors, we have that $\mathscr{M}_n$ is dense in $\mathscr{X}_n$. Indeed, given $x = \sum_{i=1}^n \alpha_i e_i$ we find sequences in $\mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$, $\left\{\alpha_{i,k}\right\}_{i=1,\ldots,n,\ k\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} \alpha_{i,k} = \alpha_i.$$

The sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathscr{M}_n$ defined by $x_n = \sum_{i=1}^n \alpha_{i,k} e_i$ will then converge to x as

$$\|x_n - x\| \leq \sum_{i=1}^n \left|\alpha_{i,k} - \alpha_i\right| \|e_i\| \underset{k\to\infty}{\longrightarrow} 0.$$

The fact that $\operatorname{span} M = \cup_{n\in\mathbb{N}} \mathscr{X}_n$ together with the density of $\mathscr{M}_n$ in $\mathscr{X}_n$ implies that $\mathscr{M}$ is dense in $\operatorname{span} M$. Indeed, let $x \in \operatorname{span} M$. Then, there exists $n_0 \in \mathbb{N}$ such that $x \in \mathscr{X}_{n_0}$. Consequently we can find a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathscr{M}_{n_0} \subset \mathscr{M}$ that converges to x showing the density of $\mathscr{M}$ in $\operatorname{span} M$.
What we have is enough to show the density of $\mathscr{M}$ in $\mathscr{X}$, which will imply its separability. To do so we will revert back to epsilons: Given $x \in \mathscr{X}$ and

$\varepsilon > 0$ we can find $x_\varepsilon \in \operatorname{span} M$ such that $\|x - x_\varepsilon\| < \frac{\varepsilon}{2}$. Since $\mathscr{M}$ is dense in $\operatorname{span} M$ we can fine $m_\varepsilon \in \mathscr{M}$ such that $\|x_\varepsilon - m_\varepsilon\| < \frac{\varepsilon}{2}$. Thus

$$\|x - m_\varepsilon\| \leq \|x - x_\varepsilon\| + \|x_\varepsilon - m_\varepsilon\| < \varepsilon.$$

The fact that x and ε were arbitrary show the desired density.

Solution to Question 8. From a theorem from class it is enough to show that $\ell_\infty(\mathbb{N})$ is not separable. Consider the family of vectors

$$\mathscr{D} = \{\boldsymbol{d} = (d_1, d_2, \dots) \mid d_n = 0 \text{ or } 1,\ \ n \in \mathbb{N}\}$$

If $\boldsymbol{d}_1 \neq \boldsymbol{d}_2$ there must be an index $n_0 \in \mathbb{N}$ such that $d_{1,n_0} \neq d_{2,n_0}$ which imply that

$$\|\boldsymbol{d}_1 - \boldsymbol{d}_2\| = \sup_{n\in\mathbb{N}} \left|d_{1,n} - d_{2,n}\right| \geq \left|d_{1,n_0} - d_{2,n_0}\right| = 1$$

(in fact, we have that $\|\boldsymbol{d}_1 - \boldsymbol{d}_2\| = 1$). Since the cardinality of $\mathscr{D}$ is $2^{\mathbb{N}}$, which is uncountable, we conclude that $\ell_\infty(\mathbb{N})$ is not separable due to another theorem from class.

Solution to Question 9. Denoting by $\boldsymbol{e}_n = e^{inx}$ where $n \in \mathbb{Z}$ we know that due to Parseval's identity we have that

$$\sum_{n\in\mathbb{Z}} \left|\langle f, \boldsymbol{e}_n\rangle\right|^2 = \|f\|^2_{L^2[-\pi,\pi]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}.$$

On the other hand we find that

$$\langle f, \boldsymbol{e}_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx$$

$$= \begin{cases} 0 & n = 0 \\ -\frac{xe^{-inx}}{2\pi i n}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi i n}\underbrace{\int_{-\pi}^{\pi} e^{-inx} dx}_{=0} & n \neq 0 \end{cases} = \begin{cases} 0 & n = 0 \\ \frac{(-1)^{n+1}}{in} & n \neq 0 \end{cases}.$$

Thus

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{1}{2}\sum_{n\in\mathbb{Z},\, n\neq 0} \left|\frac{(-1)^{n+1}}{in}\right|^2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left|\langle f, \boldsymbol{e}_n\rangle\right|^2 = \frac{\pi^2}{6}.$$

Solution to Question 10. Assume that $\mathscr{B} = \{e_\alpha\}_{\alpha\in\mathscr{I}}$ is an orthonormal basis. Since $\mathscr{H}$ is infinite dimensional we know that $\mathscr{B}$ has a countable subset. Let $\mathscr{B}_1 = \{e_n\}_{n\in\mathbb{N}}$ be such subset. Consider the vector

$$x = \sum_{n\in\mathbb{N}} \frac{1}{n} e_n.$$

Since $\sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$ we know that x is a well defined vector in $\mathscr{H}$ and that the above is its basis representation. Indeed, defining

$$S_N = \sum_{n=1}^{N} \frac{1}{n} e_n$$

we find, by Pythagoras's theorem, that

$$\| S_N - S_M \|^2 = \sum_{\min(N,M)+1}^{\max(N,M)} \frac{1}{n^2}.$$

Since $\sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$ we find that $\{S_N\}_{N\in\mathbb{N}}$ is Cauchy and since the space is complete it must converge.
Moreover, since

$$\langle x, e_\alpha \rangle \neq 0$$

for an infinite set of vectors from our orthonormal set we *can't* find a finite set $\mathscr{F} = \{e_{\alpha_1}, \dots, e_{\alpha_n}\}$ such that $x \in \text{span}\mathscr{F}$. Thus $\mathscr{B}$ can't be a Hamel basis.
To show the second statement we notice that if $\mathscr{H}$ has a countable Hamel basis $\mathscr{B} = \{x_n\}_{n\in\mathbb{N}}$ then by the process of the Gran-Schmidt procedure we would have found a countable orthonormal basis that is a Hamel basis. Indeed, for any $n \in \mathbb{N}$ we can find $k(n) \in \mathbb{N}$ such that the orthonormal set $\{e_1, \dots, e_{k(n)}\}$ satisfies

$$\text{span}\{x_1, \dots, x_n\} = \text{span}\{e_1, \dots, e_{k(n)}\}.$$

This contradicts the first part of the problem, giving us the desired result.

Solution to Question 11. For a given $x \in \mathscr{H}$ and $k \in \mathbb{N}$ we define the set

$$M_k(x) = \left\{ i \in \mathscr{I} \mid |\langle x, e_i \rangle| \geq \frac{1}{k} \right\}.$$

We claim that $M_k(x)$ must be finite. Indeed, if we can find a sequence $\{i_n\}_{n\in\mathbb{N}} \subset \mathscr{I}$ such that $\left|\langle x, e_{i_n} \rangle\right| \geq \frac{1}{k}$ then

$$\sum_{n\in\mathbb{N}} \left|\langle x, e_{i_n} \rangle\right|^2 \geq \sum_{n\in\mathbb{N}} \frac{1}{k} = \infty.$$

However, since $\widetilde{\mathscr{B}} = \{e_{i_n}\}_{n\in\mathbb{N}}$ is orthonormal, the above contradicts Bessel's inequality.
To conclude the proof we notice that

$$\langle x, e_i \rangle \neq 0 \quad \Leftrightarrow \quad |\langle x, e_i \rangle| \geq \frac{1}{k} \text{ for some } k \in \mathbb{N} \quad \Leftrightarrow \quad i \in \cup_{k\in\mathbb{N}} M_k(x).$$

As $M(x) = \cup_{k\in\mathbb{N}} M_k(x)$ is a countable union of finite sets, it is countable, and we just showed that for any $i \notin M(x)$ we must have that $\langle x, e_i \rangle = 0$.

Solution to Question 12. We start by claiming that $\mathscr{H}^{\perp} = \{0\}$. Indeed, if $y \in \mathscr{H}^{\perp}$ then since y is also in $\mathscr{H}$ we find that

$$0 = \langle y, y \rangle = \|y\|^2.$$

This implies that $y = 0$, showing that $\mathscr{H}^{\perp} = \{0\}$.
Assume now that $\mathscr{M}^{\perp} = \{0\}$. Since $\mathscr{M}$ is closed we know from a previous question that

$$\mathscr{M} = \mathscr{M}^{\perp\perp} = \{0\}^{\perp} = \mathscr{H}$$

where the last identity follows from the fact that every vector is perpendicular to the zero vector.

Solution to Question 13. Let $\mathcal{M}$ be the set of all orthonormal sets in $\mathscr{H}$. $\mathcal{M} \neq \emptyset$ since $\mathscr{H}$ is not the trivial vector space. We define a partial order on $\mathcal{M}$ by inclusion and claim that if $\mathcal{M}$ has a maximal element, $\mathscr{B}_{\max}$, then $\mathscr{H} = \overline{\text{span}\mathscr{B}_{\max}}$ which, according to a theorem from class, implies that $\mathscr{B}_{\max}$ is an orthonormal basis for $\mathscr{H}$.
Indeed, if this is not the case then we can find some x in $\mathscr{H} \setminus \widetilde{\mathscr{H}}$, where $\widetilde{\mathscr{H}} = \overline{\text{span}\mathscr{B}_{\max}}$. Since $\widetilde{\mathscr{H}}$ is a closed subspace of $\mathscr{H}$ the vector $P_{\widetilde{\mathscr{H}}}x \in \widetilde{\mathscr{H}}$ is well defined and $v = x - P_{\widetilde{\mathscr{H}}}(x)$ is a non-zero vector in $\widetilde{\mathscr{H}}^{\perp}$. Consequently, the set

$$\widetilde{\mathscr{B}} = \mathscr{B}_{\max} \cup \left\{ \frac{v}{\|v\|} \right\}$$

is an orthonormal set that is larger than $\mathscr{B}_{\max}$, which is a contradiction.
To show that we have a maximal element we will invoke Zorn's lemma. In order to do that we will need to show that the conditions of the lemma hold, i.e. that every chain in $\mathcal{M}$ has an upper bound.
Let $\mathcal{C}$ be a chain in $\mathcal{M}$ and define

$$U = \cup_{A \in \mathcal{C}} A.$$

We claim that U is orthonormal. Indeed, if $x_1, x_2 \in U$ then there exist $A_1,, A_2 \in \mathcal{C}$ such that $x_1 \in A_1$ and $x_2 \in A_2$. Since A_1 and A_2 are in a chain, one of these sets contains the other. Without loss of generality $A_2 \subseteq A_1$. Thus, $x_1, x_2 \in A_1$, and since A_1 is an orthonormal set we conclude that x_1 and x_2 are of norm 1 and are orthogonal. As x_1 and x_2 were arbitrary we conclude that U is orthonormal and as such in $\mathcal{M}$. By the definition of the partial order of $\mathcal{M}$ we have that $A \leq U$ for any $A \in \mathcal{C}$. We conclude that every chain in $\mathcal{M}$ have an upper bound and conclude the proof.