Home Assignment 3

Exercise 1. Prove the following statement: The condition

$$\|x\|_2 \leq c\,\|x\|_1 \tag{1}$$

for some $c > 0$ and all $x \in \mathcal{X}$ is equivalent to any (and all) of the following:

(i) For any $x \in \mathcal{X}$ with $\|x\|_2 > 1$ we have that $\|x\|_1 > \frac{1}{c}$.
(ii) For any $x \in \mathcal{X}$ with $\|x\|_2 \geq 1$ we have that $\|x\|_1 \geq \frac{1}{c}$.
(iii) For any $x \in \mathcal{X}$ with $\|x\|_2 = 1$ we have that $\|x\|_1 \geq \frac{1}{c}$

Exercise 2. Prove the following statement: Let $\mathcal{X}$ be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathcal{X}$. Then if the topologies of the normed spaces $(\mathcal{X}, \|\cdot\|_1)$ and $(\mathcal{X}, \|\cdot\|_2)$ are equivalent we have that:

(i) The sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ converges to $x \in \mathcal{X}$ in $(\mathcal{X}, \|\cdot\|_1)$ if and only if it converges to x in $(\mathcal{X}, \|\cdot\|_2)$.
(ii) The sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ is Cauchy in $(\mathcal{X}, \|\cdot\|_1)$ if and only if it Cauchy in $(\mathcal{X}, \|\cdot\|_2)$.
(iii) $(\mathcal{X}, \|\cdot\|_1)$ is a Banach space if and only if $(\mathcal{X}, \|\cdot\|_2)$ is.
(iv) Given an additional normed space $\mathcal{Y}$ we have that $f : \mathcal{Y} \to (\mathcal{X}, \|\cdot\|_1)$ is continuous if and only if $f : \mathcal{Y} \to (\mathcal{X}, \|\cdot\|_2)$ is.

Exercise 3. Show that for any $1 \leq p, q \leq \infty$ the norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max_{i=1,\dots,n} |x_i| & p = \infty \end{cases}$$

are equivalent on $\mathbb{R}^n$ or $\mathbb{C}^n$.
Hint: You may use without proof the following discrete Hölder inequality:

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^q\right)^{\frac{1}{q}}.$$

where $p, q \in (1, \infty)$ are Hölder conjugates.

Exercise 4. Consider the space $\ell_p(\mathbb{N})$ for some $1 \leq p < \infty$. Show that $\ell_p(\mathbb{N}) \subset_{\neq} \ell_\infty(\mathbb{N})$ and conclude that the norm of $\ell_\infty(\mathbb{N})$, $\|\cdot\|_\infty$, is a norm on $\ell_p(\mathbb{N})$. Show that this norm is not equivalent to the standard norm $\|\cdot\|_p$.

Exercise 5. Prove the following statement: The notion of equivalence of norms is transitive, i.e. if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ and $\|\cdot\|_2$ is equivalent to $\|\cdot\|_3$ then $\|\cdot\|_1$ is equivalent to $\|\cdot\|_3$

Exercise 6. Prove the following statement: Let $\{e_1,\ldots,e_n\}$ be a basis for a vector space $\mathscr{X}$. For a given $x\in\mathscr{X}$ let $\{\alpha_i(x)\}_{i=1,\ldots,n}$ be the unique scalars such that

$$x=\sum_{i=1}^{n}\alpha_i(x)e_i.$$

Show that for any $x\in\mathscr{X}$ and any scalar β we have that

$$\alpha_i\left(\beta x\right)=\beta\alpha_i(x),$$

and for any $x,y\in\mathscr{X}$

$$\alpha_i\left(x+y\right)=\alpha_i(x)+\alpha_i(y).$$

Exercise 7. Prove the following statement: Any finite dimensional normed space $(\mathscr{X},\|\cdot\|)$ is complete. Consequently, any finite dimensional subspace $\mathscr{M}$ of a Banach space $\mathscr{X}$ is closed.

Exercise 8. Show an improved version of F. Riesz's lemma in the special case where the space is an inner product space: Let $\mathscr{H}$ be an inner product subspace and let $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. If $\mathscr{M}\neq\mathscr{H}$ then there exists $x\in\mathscr{H}$ of norm 1 such that

$$\inf_{y\in\mathscr{M}}\left\|x-y\right\|=1.$$

Exercise 9. Let $\mathscr{X}$ and $\mathscr{Y}$ be vector spaces and assume that $T:\mathscr{X}\to\mathscr{Y}$ is linear and a bijection. Show that T^{-1} is also a linear operator.

Exercise 10. Consider the multiplication operator $M:(C[a,b],\|\cdot\|_\infty)\to(C[a,b],\|\cdot\|_\infty)$ defined by

$$Mf(x)=m(x)f(x)$$

where $m\in C[a,b]$. Show that M is a linear bounded operator.

Exercise 11. Consider the kernel operator $K:(C[a,b],\|\cdot\|_\infty)\to(C[a,b],\|\cdot\|_\infty)$ defined by

$$Kf(x)=\int_a^b k(x,y)f(y)dy,$$

where $k\left(x,y\right)\in C([a,b]\times[a,b])$. Show that K is a linear bounded operator.

Exercise 12. Prove the following statement: Let $\mathscr{X}$ and $\mathscr{Y}$ be two normed spaces and let $T:\mathscr{D}(T)\subseteq\mathscr{X}\to\mathscr{Y}$ be a linear operator. If $\dim\mathscr{D}(T)<\infty$ then T is bounded.

Exercise 13. Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces and let $T\in L\left(\mathscr{X},\mathscr{Y}\right)$. Show that T is bounded if and only if T maps bounded sets into bounded sets.

Exercise 14. Prove the following statement: Let $\mathscr{X}$ and $\mathscr{Y}$ be two normed spaces and let $T : \mathscr{D}(T) \subset \mathscr{X} \to \mathscr{Y}$ be a bounded linear operator. If $\mathscr{Y}$ is a Banach space, then T can be uniquely extended to $\overline{\mathscr{D}(T)}$, i.e. there exists a unique linear bounded operator $\widetilde{T} : \overline{\mathscr{D}(T)} \subset \mathscr{X} \to \mathscr{Y}$ such that $\widetilde{T}|_{\mathscr{D}(T)} = T$. Moreover

$$\sup_{x \in \overline{\mathscr{D}(T)},\, x \neq 0} \frac{\|\widetilde{T}x\|}{\|x\|} = \sup_{x \in \mathscr{D}(T),\, x \neq 0} \frac{\|Tx\|}{\|x\|}.$$