Solution to Home Assignment 3

Solution to Question 1. Much like in class, it is immediate to notice that (1) implies $(i)-(iii)$. Since $\|x\|=1$ implies $\|x\|\geq 1$ it is also clear that (ii) implies (iii). In order to conclude the desired result we will show that (i) implies (ii) and that (iii) implies (1).
Assume that (i) holds and let $x\in\mathscr{X}$ be such that $\|x\|_2\geq 1$. For any $0<\varepsilon<1$ define $x_\varepsilon=\frac{x}{1-\varepsilon}$. We find that

$$\|x_\varepsilon\|_2=\frac{\|x\|_2}{1-\varepsilon}>1,$$

and as (i) holds we find that

$$\frac{\|x\|_1}{1-\varepsilon}=\|x_\varepsilon\|_1>\frac{1}{c}.$$

As the above holds for all $0<\varepsilon<1$, taking ε to zero yields the inequality $\|x\|_1\geq\frac{1}{c}$.
Next we assume that (iii) hold and consider $x\in\mathscr{X}$. If $x=0$ then (1) holds trivially for any $c>0$. We can assume, therefore, that $x\neq 0$ and define $y_x=\frac{x}{\|x\|_2}$. We have that $\left\|y_x\right\|_2=1$ and as (iii) holds we conclude that

$$\frac{\|x\|_1}{\|x\|_2}=\left\|y_x\right\|_1\geq\frac{1}{c}$$

which can be re-arranged to be $\|x\|_2\leq c\,\|x\|_1$. Together with the case $x=0$ we conclude the proof.

Solution to Question 2. (i) Let us start by assuming that there exists $c>0$ such that for any $x\in\mathscr{X}$ we have that

$$\|x\|_2\leq c\,\|x\|_1$$

and let $\{x_n\}_{n\in\mathbb{N}}\subset\mathscr{X}$ be a sequences that converges to $x\in\mathscr{X}$ in $\|\cdot\|_1$. Then by the pinching lemma we have that

$$0\leq\|x-x_n\|_2\leq c\,\|x-x_n\|_1$$

which shows that $\{x_n\}_{n\in\mathbb{N}}\subset\mathscr{X}$ converges to $x\in\mathscr{X}$ in $\|\cdot\|_2$. A symmetric argument shows that if there exists $c>0$ such that

$$\|x\|_2\geq\frac{1}{c}\,\|x\|_1$$

then if $\{x_n\}_{n\in\mathbb{N}}\subset\mathscr{X}$ converges to $x\in\mathscr{X}$ in $\|\cdot\|_2$ then it must also converge in $\|\cdot\|_1$. Combing the above with the fact that the topologies of normed spaces are equal if and only if there exists $c_1,c_2>0$ such that

$$\frac{\|x\|_1}{c}\leq\|x\|_2\leq c\,\|x\|_1$$

shows the first statement.

(ii) As we see from the above solution, due to the symmetry of the condition, it is enough for us to show that the condition

$$\|x\|_2 \leq c\,\|x\|_1$$

implies that if $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ is Cauchy in $\|\cdot\|_1$ then it must be Cauchy in $\|\cdot\|_2$ as well. Indeed, given $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for any $n, m \geq n_0$ we have that

$$\|x_n - x_m\|_1 < \frac{\varepsilon}{c}.$$

Consequently, for any $n, m \geq n_0$

$$\|x_n - x_m\|_2 \leq c\,\|x_n - x_m\|_1 < \varepsilon.$$

(iii) Assume that $(\mathscr{X}, \|\cdot\|_1)$ is Banach and let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\|\cdot\|_2$. According to the previous sub-question we know that as the topologies are equivalent $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $\|\cdot\|_1$. Since $(\mathscr{X}, \|\cdot\|_1)$ is Banach, there exists $x \in \mathscr{X}$ such that $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ converges to $x \in \mathscr{X}$ in $\|\cdot\|_1$. As was seen in the first sub-question this implies that $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ converges to $x \in \mathscr{X}$ in $\|\cdot\|_2$, which shows that $(\mathscr{X}, \|\cdot\|_2)$ is a Banach space. Replacing $\|\cdot\|_1$ with $\|\cdot\|_2$ proves the equivalence.

(iv) Again, it is enough to show only on direction. Let $f : (\mathscr{X}, \|\cdot\|_1) \to \mathscr{Y}$ be continuous and assume that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $\|\cdot\|_2$. According to the first sub-question we know that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in $\|\cdot\|_1$. As f is continuous under that topology we know that $\lim_{n\to\infty} f(x_n) = f(x)$, which shows the continuity under $\|\cdot\|_2$.

Solution to Question 3. We start with the case $1 < p < \infty$ and $q = \infty$. By definition

$$\max_{i=1,\dots,n} |x_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} 1\right)^{\frac{1}{p}} \max_{i=1,\dots,n} |x_i|$$

which shows that

$$\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}}\,\|x\|_\infty. \tag{1}$$

The above inequalities are sharp as the standard basis vector $\boldsymbol{e}_1 = (1, 0, \dots, 0)$ (or any other standard basis choice) gives equality on the left hand side and the vector $a = (1, 1, \dots, 1)$ gives equality on the right hand side.

Next, we consider the case $1 < p < q < \infty$. Using the discrete Hölder inequality we find that with the choice $r = \frac{q}{p}$ and its Hölder conjugate $r' = \frac{q}{q-p}$

$$\sum_{i=1}^{n} |x_i|^p \leq \left(\sum_{i=1}^{n} |x_i|^{rp}\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} 1^{r'}\right)^{\frac{1}{r'}}$$

from which we get that

$$\|x\|_p \leq n^{\frac{q-p}{pq}} \|x\|_q .$$

In addition,

$$\sum_{i=1}^n |x_i|^q \leq \|x\|_\infty^{q-p} \sum_{i=1}^n |x_i|^p \leq \|x\|_p^{q-p} \left(\sum_{i=1}^n |x_i|^p \right) = \|x\|_p^q .$$

Adding these two observations we conclude that in this case

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{q-p}{pq}} \|x\|_q . \tag{2}$$

These inequalities are, again, sharp by choosing $\boldsymbol{e}_1$ and a again.

Solution to Question 4. We start by noticing that if $\boldsymbol{a} \in \ell_p(\mathbb{N})$ for some $1 \leq p < \infty$ then $\lim_{n\to\infty} a_n = 0$, and as such the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded, i.e. belongs to $\ell_\infty(\mathbb{N})$. Since the addition and scalar multiplication that is defined in all of $\ell_p(\mathbb{N})$ −s, $1 \leq p \leq \infty$, is identical we conclude that $\ell_p(\mathbb{N})$ is a subspace of $\ell_\infty(\mathbb{N})$ and as such we can consider the induced/restricted norm $\|\cdot\|_\infty$ on it.
Next, we notice that for any $n \in \mathbb{N}$

$$|a_n| \leq \left(\sum_{k\in\mathbb{N}} |a_k|^p \right)^{\frac{1}{p}} = \|\boldsymbol{a}\|_p .$$

Taking the supremum over $n \in \mathbb{N}$ gives us

$$\|\boldsymbol{a}\|_\infty \leq \|\boldsymbol{a}\|_p .$$

The converse, however, doesn't hold. Indeed, consider the vectors

$$\boldsymbol{a}_n = \sum_{i=1}^n \boldsymbol{e}_i = \Big(1, 1, \dots, \underbrace{1}_{n-\text{th position}}, 0, \dots \Big).$$

we see that

$$\|\boldsymbol{a}_n\|_\infty = 1$$
$$\|\boldsymbol{a}_n\|_p = n^{\frac{1}{p}},$$

which shows that there can't be a constant $c > 0$ such that

$$\|\boldsymbol{a}\| \leq c \|\boldsymbol{a}\|_\infty$$

for all $\boldsymbol{a} \in \ell_p(\mathbb{N})$ since that would have implied that

$$n^{\frac{1}{p}} = \|\boldsymbol{a}_n\|_p \leq c \|\boldsymbol{a}_n\|_\infty = c$$

for all $n \in \mathbb{N}$. Consequently the norm $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are not equivalent on $\ell_p(\mathbb{N})$.

Solution to Question 5. We assume that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ and $\|\cdot\|_2$ is equivalent to $\|\cdot\|_3$. This implies that we can find constants $c > 0$ and $d > 0$ such that

$$\frac{\|x\|_1}{c} \leq \|x\|_2 \leq c\,\|x\|_1\,,$$

and

$$\frac{\|x\|_2}{d} \leq \|x\|_3 \leq d\,\|x\|_2\,.$$

Combining the above we see that

$$\frac{\|x\|_1}{cd} \leq \|x\|_3 \leq cd\,\|x\|_1\,,$$

which shows that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_3$.

Solution to Question 6. Let $x \in \mathscr{X}$ and $\beta \in \mathbb{F}$ be given and assume that $\{\alpha_i(x)\}_{i=1,\ldots,n}$ are the *unique* scalars such that

$$x = \sum_{i=1}^{n} \alpha_i(x) e_i.$$

Then, by definition,

$$\beta x = \sum_{i=1}^{n} \alpha_i\left(\beta x\right) e_i.$$

On the other hand,

$$\beta x = \beta\left(\sum_{i=1}^{n} \alpha_i(x) e_i\right) = \sum_{i=1}^{n} \left(\beta\alpha_i(x)\right) e_i.$$

From the uniqueness of the coefficients we must have that $\alpha_i\left(\beta x\right) = \beta\alpha_i(x)$ for all $i = 1, \ldots, n$.
Similarly, for any $x, y \in \mathscr{X}$ we have that

$$x + y = \sum_{i=1}^{n} \alpha_i(x+y) e_i$$

and

$$x + y = \sum_{i=1}^{n} \alpha_i(x) e_i + \sum_{i=1}^{n} \alpha_i(y) e_i = \sum_{i=1}^{n} \left(\alpha_i(x) + \alpha_i(y)\right) e_i,$$

which implies, due to the uniqueness of the coefficients again, that

$$\alpha_i(x+y) = \alpha_i(x) + \alpha_i(y)$$

for all $i = 1, \ldots, n$.

Solution to Question 7. Let $\mathscr{X}$ be a finite dimensional normed space. From a previous question we know that if if two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, are equivalent then $(\mathscr{X},\|\cdot\|_1)$ is a Banach space if and only if $(\mathscr{X},\|\cdot\|_2)$ is, and since we know from class that all the norms are equivalent on a finite dimensional normed space, we conclude that it is enough for us to find *a* norm on $\mathscr{X}$ under which it is a Banach space. We will consider the Euclidean-like norm, defined in class:

$$\|x\|_{\text{Euclid}} = \sqrt{\sum_{i=1}^{d} |\alpha_i(x)|^2}$$

where $x = \sum_{i=1}^{d} \alpha_i(x) e_i$ is the unique representation of x with respect to a basis $\{e_1,\dots,e_d\}$ of $\mathscr{X}$. We know that it is a norm on $\mathscr{X}$.
Consider a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$. Since for any $j \in \{1,\dots,d\}$

$$\left|\alpha_j(x_n) - \alpha_j(x_m)\right| = \left|\alpha_j(x_n - x_m)\right| \leq \sqrt{\sum_{i=1}^{d} \left|\alpha_j(x_n - x_m)\right|^2}$$

$$= \sqrt{\sum_{i=1}^{d} \left|\alpha_j(x_n) - \alpha_i(x_m)\right|^2} = \|x_n - x_m\|_{\text{Euclid}}$$

we see that $\{\alpha_i(x_n)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{F}$ for any $i = 1,\dots,d$ and consequently it must converge to a scalar α_i. We define

$$x = \sum_{i=1}^{d} \alpha_i e_i \in \mathscr{X}$$

and notice that

$$\|x_n - x\|_{\text{Euclid}} = \sqrt{\sum_{i=1}^{d} |\alpha_i(x_n) - \alpha_i|^2} \underset{n\to\infty}{\longrightarrow} 0.$$

This implies that $(\mathscr{X},\|\cdot\|_{\text{Euclid}})$ is a Banach space, which is what we wanted to show.

Solution to Question 8. Let $\mathscr{M}$ be a closed subspace of $\mathscr{H}$ that is not $\mathscr{H}$ and let $v \in \mathscr{H}\setminus\mathscr{M}$. Since $\mathscr{M}$ is a closed subspace of $\mathscr{H}$, $P_{\mathscr{M}}v$ is a well defined vector in $\mathscr{M}$, and since $v \notin \mathscr{M}$ we have that $v \neq P_{\mathscr{M}}v$. Moreover, $v - P_{\mathscr{M}}v \perp \mathscr{M}$. Define

$$x = \frac{v - P_{\mathscr{M}}v}{\|v - P_{\mathscr{M}}v\|}.$$

We find that $\|x\| = 1$ and since $x \perp \mathscr{M}$

$$\left\|x - y\right\| = \sqrt{1 + \left\|y\right\|^2} \geq 1,$$

for any $y \in \mathcal{M}$. We have that

$$\inf_{y\in\mathcal{M}} \left\|x-y\right\| \geq 1.$$

Since, in addition,

$$\inf_{y\in\mathcal{M}} \left\|x-y\right\| \leq \|x-0\| = 1$$

we conclude that

$$\inf_{y\in\mathcal{M}} \left\|x-y\right\| = 1.$$

Solution to Question 9. Since T is a bijection from $\mathcal{X}$ to $\mathcal{Y}$ we know that the map $T^{-1}: \mathcal{Y} \to \mathcal{X}$ exists and satisfies

$$TT^{-1} = Id_{\mathcal{Y}}, \qquad T^{-1}T = Id_{\mathcal{X}}.$$

Let $y_1, y_2 \in \mathcal{Y}$. We have that

$$y_1 + y_2 = T\left(T^{-1}\left(y_1+y_2\right)\right).$$

On the other hand

$$y_1 + y_2 = T\left(T^{-1}y_1\right) + T\left(T^{-1}y_2\right) \underset{T\in L(\mathcal{X},\mathcal{Y})}{=} T\left(T^{-1}y_1 + T^{-1}y_2\right).$$

We conclude that

$$T\left(T^{-1}\left(y_1+y_2\right)\right) = T\left(T^{-1}y_1 + T^{-1}y_2\right),$$

which implies, due to the injectivity of T, that

$$T^{-1}\left(y_1+y_2\right) = T^{-1}y_1 + T^{-1}y_2.$$

Similarly, for any $x \in \mathcal{X}$ and a scalar β we have that

$$T^{-1}\left(\beta Tx\right) \underset{T\in L(\mathcal{X},\mathcal{Y})}{=} T^{-1}\left(T\left(\beta x\right)\right) = \beta x = \beta T^{-1}Tx.$$

For a given $y \in \mathcal{Y}$ we can find $x_y \in \mathcal{X}$ such that $Tx_y = y$. Using the above we conclude that

$$T^{-1}\left(\beta y\right) = \beta x_y = \beta T^{-1}y$$

which concludes the linearity of T^{-1}.

Solution to Question 10. The linearity follows from the fact that for any functions f and g

$$m(x)\left(f(x)+g(x)\right) = m(x)f(x) + m(x)g(x),$$

and for any function f and scalar α

$$m(x)\left(\alpha f(x)\right) = \alpha m(x)f(x).$$

The boundedness follows from the fact that for any $f \in C[a,b]$

$$\left|m(x)f(x)\right| = |m(x)|\left|f(x)\right| \leq \left(\max_{z\in[a,b]}|m(z)|\right)\left|f(x)\right| = \|m\|_{\infty}\left|f(x)\right|,$$

which implies that for any $f \in C[a,b]$

$$\|Mf\|_\infty = \max_{x\in[a,b]} |Mf(x)| \le \|m\|_\infty \max_{x\in[a,b]} |f(x)| = \|m\|_\infty \|f\|_\infty .$$

Solution to Question 11. We start by noticing that since $k \in C([a,b]\times[a,b])$ and $f \in C[a,b]$ we have that $k(x,y)f(x) \in C([a,b]\times[a,b])$ and as such integrable. Known results from Analysis show that $Kf \in C[a,b]$. Let us prove it:

$$|Kf(x) - Kf(y)| = \left|\int_a^b \left(k(x,z)-k(y,z)\right) f(z)dz\right|$$

$$\le \int_a^b |k(x,z)-k(y,z)|\,|f(z)|\,dz \le \sup_{z\in[a,b]} |k(x,z)-k(y,z)| \int_a^b |f(z)|\,dz.$$

Since $k \in C([a,b]\times[a,b])$ and $[a,b]\times[a,b]$ is compact, k is *uniformly continuous on it*, meaning that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sqrt{(x_1-x_2)^2 + \left(y_1-y_2\right)^2} < \delta$ then

$$\left|k\left(x_1,y_1\right) - k\left(x_2,y_2\right)\right| < \varepsilon.$$

Thus, if $|x-y| < \delta$ we find that $\sqrt{\left(x-y\right)^2 + (z-z)^2} < \delta$ and

$$|Kf(x) - Kf(y)| \le \varepsilon \underbrace{\int_a^b |f(z)|\,dz}_{\text{fixed constant}},$$

which shows the continuity.
Next we show the linearity: Given $f_1, f_2 \in C[a,b]$ and a scalar α we see from the properties of integration that

$$K\left(f_1+f_2\right)(x) = \int_a^b k(x,y)\left(f_1(y)+f_2(y)\right)dy = \int_a^b k(x,y)f_1(y)dy$$
$$+\int_a^b k(x,y)f_2(y)dy = Kf_1(x) + Kf_2(x)$$

and

$$K\left(\alpha f\right)(x) = \int_a^b k(x,y)\left(\alpha f\right)(y)dy = \alpha\int_a^b k(x,y)f(y)dy = \alpha Kf(x),$$

which shows the desired properties.
Lastly we notice that for any $x \in [a,b]$

$$|Kf(x)| = \left|\int_a^b k(x,y)f(y)dy\right| \le \int_a^b |k(x,y)|\,|f(y)|\,dy$$

$$\le \|f\|_\infty \int_a^b |k(x,y)|\,dy.$$

Thus

$$\|Kf\|_\infty \le \sup_{x\in[a,b]}\left(\int_a^b \left|k(x,y)\right| dy\right)\|f\|_\infty = \left(\max_{x\in[a,b]}\int_a^b \left|k(x,y)\right| dy\right)\|f\|_\infty,$$

from which we conclude the boundedness of K. Note that the last equality holds due to the fact that $\int_a^b \left|k(x,y)\right| dy$ is a continuous function of x.

Solution to Question 12. Since $\mathscr{D}(T)$ is finite dimensional we can find a basis for it, $\{e_1,\dots,e_n\}$. Any $x\in\mathscr{X}$ can be written uniquely as $x=\sum_{i=1}^n \alpha_i(x)e_i$. Consequently, any linear operator $T:\mathscr{D}(T)\to\mathscr{Y}$ satisfies

$$\|Tx\| = \left\|\sum_{i=1}^n \alpha_i(x)Te_i\right\| \le \sum_{i=1}^n |\alpha_i(x)|\,\|Te_i\| \le \sqrt{\sum_{i=1}^n |\alpha_i(x)|^2}\sqrt{\sum_{i=1}^n \|Te_i\|^2}.$$

We know that as $\mathscr{D}(T)$ is finite dimensional all the norms on it are equivalent. Much like in the lectures we know that

$$\|x\|_2 = \left\|\sum_{i=1}^n \alpha_i(x)e_i\right\|_2 = \sqrt{\sum_{i=1} |\alpha_i(x)|^2}$$

is a norm on $\mathscr{X}$ and as such there there exists $c>0$ such that

$$\|x\|_2 \le c\,\|x\|.$$

Consequently

$$\|Tx\| \le \|x\|_2\sqrt{\sum_{i=1}^n \|Te_i\|^2} \le c\sqrt{\sum_{i=1}^n \|Te_i\|^2}\,\|x\|,$$

which shows that T is bounded.

Solution to Question 13. We start by assuming that T is bounded. Let B be a bounded set. Then, there exists $M>0$ such that $\|x\|\le M$ for any $x\in B$. Since T is bounded there exists $C>0$ such that for any $x\in\mathscr{X}$

$$\|Tx\| \le C\,\|x\|$$

and consequently

$$\sup_{x\in B}\|Tx\| \le \sup_{x\in B} C\,\|x\| \le CM$$

which shows one direction.
Let us now assume that T takes bounded sets to bounded sets. This implies that $\sup_{x\in\mathscr{X},\,\|x\|=1}\|Tx\|<\infty$ and consequently, as we saw in class, T is bounded.
We enclose the proof: Consider the set $B=\{x\in\mathscr{X}\mid \|x\|=1\}$. As B is bounded we know that there exists $C>0$ such that $\sup_{x\in B}\|Tx\|=C<\infty$. We claim that for any $x\in\mathscr{X}$ we have that

$$\|Tx\| \le C\,\|x\|$$

which will show the boundedness. As the above holds trivially for $x = 0$ we can assume without loss of generality that $x \neq 0$. Defining $y_x = \frac{x}{\|x\|}$ we find that $\|y_x\| = 1$ and thus

$$\frac{\|Tx\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|}\right)\right\| = \|Ty\| \leq C$$

which gives us the desired inequality.

Solution to Question 14. Let $x \in \overline{\mathscr{D}(T)} \setminus \mathscr{D}(T)$ be given and let $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ be a sequence that converges to x. Since every converging sequence is Cauchy and since T is bounded we see that for any $n, m \in \mathbb{N}$

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq C\|x_n - x_m\|.$$

We conclude that $\{Tx_n\}_{n\in\mathbb{N}}$ is also Cauchy (we saw this argument in a previous question) and since $\mathscr{Y}$ is a Banach space there exists $y \in \mathscr{Y}$ such that $\lim_{n\to\infty} Tx_n = y$. We would like to define $\widetilde{T}x = y$, but in order to do that we *must show* that the limit we found doesn't depend on the choice of the sequence $\{x_n\}_{n\in\mathbb{N}}$.
Indeed, assume that $\{x_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ both converge to x. According to what we've just shown, there exist y_x and y_z in $\mathscr{Y}$ such that

$$Tx_n \underset{n\to\infty}{\longrightarrow} y_x, \qquad \text{and} \qquad Tx_n \underset{n\to\infty}{\longrightarrow} y_z.$$

Defining the sequence

$$\xi_n = \begin{cases} x_{2k} & n = 2k \\ z_{2k+1} & n = 2k+1 \end{cases}$$

which is also in $\mathscr{D}(T)$ and converges to x[1] and as such there exists y_ξ such that

$$T\xi_n \underset{n\to\infty}{\longrightarrow} y_\xi.$$

However

$$T\xi_{2n} = Tx_{2n} \underset{n\to\infty}{\longrightarrow} y_x$$
$$T\xi_{2n+1} = Tz_{2n+1} \underset{n\to\infty}{\longrightarrow} y_z$$

which shows that, due to the uniqueness of the limit and the fact that subsequences of a converging sequence must converge to the same limit, $y_x = y_z = y_\xi$. In other words, defining

$$\widetilde{T}x = \lim_{n\to\infty} Tx_n$$

[1]Given $\varepsilon > 0$ we find $n_1, n_2 \in \mathbb{N}$ such that if $n \geq n_1$ we have that $\|x_n - x\| < \varepsilon$ and if $n \geq n_2$ we have that $\|z_n - z\| < \varepsilon$. Then for any $n \geq \max\{n_1, n_2\}$ we have that

$$\|\xi_n - x\| \leq \max\{\|x_n - x\|, \|z_n - x\|\} < \varepsilon.$$

where $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ converges to $x \in \overline{\mathscr{D}(T)} \setminus \mathscr{D}(T)$ doesn't depend on the choice of the sequence. We define

$$\widetilde{T}x = \begin{cases} Tx, & x \in \mathscr{D}(T), \\ \lim_{n\to\infty} Tx_n, & x \in \overline{\mathscr{D}(T)} \setminus \mathscr{D}(T), \ \{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T) \text{ converges to } x. \end{cases}$$

and show that it is continuous and linear. We start by noticing that due to the continuity of T on $\mathscr{D}(T)$, if $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ converges to $x \in \mathscr{D}(T)$ then

$$\widetilde{T}x = Tx = \lim_{n\to\infty} Tx_n.$$

This implies that we can define $\widetilde{T}$ on $\overline{\mathscr{D}(T)}$ as

$$\widetilde{T}x = \lim_{n\to\infty} Tx_n, \qquad x \in \overline{\mathscr{D}(T)} \setminus \mathscr{D}(T), \ \{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T) \text{ converges to } x.$$

Consequently, for any $x, y \in \overline{\mathscr{D}(T)}$ we can find sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ such that $x_n \underset{n\to\infty}{\longrightarrow} x$ and $y_n \underset{n\to\infty}{\longrightarrow} y$, and since $x_n + y_n \underset{n\to\infty}{\longrightarrow} x + y$ we have that

$$\widetilde{T}\left(x+y\right) = \lim_{n\to\infty} T\left(x_n + y_n\right) = \lim_{n\to\infty}\left(Tx_n + Ty_n\right) = \widetilde{T}x + \widetilde{T}y.$$

Similarly, for any given scalar α we have that and $\alpha x_n \underset{n\to\infty}{\longrightarrow} \alpha x$ and as such

$$\widetilde{T}(\alpha x) = \lim_{n\to\infty} T(\alpha x_n) = \lim_{n\to\infty}(\alpha T x_n) = \alpha \widetilde{T}x,$$

which proves the linearity of the extension.
Next we focus on showing identity (2), which will imply boundedness of $\widetilde{T}$, as we saw in class. We starting by finding that

$$\sup_{x\in\overline{\mathscr{D}(T)},\, x\neq 0} \frac{\left\|\widetilde{T}x\right\|}{\|x\|} \geq \sup_{x\in\mathscr{D}(T),\, x\neq 0} \frac{\left\|\widetilde{T}x\right\|}{\|x\|} \underset{\widetilde{T}|_{\mathscr{D}(T)}=T}{=} \sup_{x\in\mathscr{D}(T),\, x\neq 0} \frac{\|Tx\|}{\|x\|}.$$

To show the reverse inequality we start by noticing that for any $x \in \mathscr{D}(T)$ (including $x = 0$)

$$\|Tx\| \leq \left(\sup_{x\in\mathscr{D}(T),\, x\neq 0} \frac{\|Tx\|}{\|x\|}\right)\|x\|.$$

Let $x \in \overline{\mathscr{D}(T)}$ and let $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ converge to x. Then due to the continuity of the norm we see that

$$\left\|\widetilde{T}x\right\| = \lim_{n\to\infty}\|Tx_n\| \leq \liminf_{n\to\infty}\left(\sup_{x\in\mathscr{D}(T),\, x\neq 0} \frac{\|Tx\|}{\|x\|}\right)\|x_n\| = \left(\sup_{x\in\mathscr{D}(T),\, x\neq 0} \frac{\|Tx\|}{\|x\|}\right)\|x\|,$$

from which we conclude that for any $x \in \overline{\mathscr{D}(T)}$ that is not zero

$$\frac{\left\|\widetilde{T}x\right\|}{\|x\|} \leq \left(\sup_{x\in\mathscr{D}(T),\, x\neq 0} \frac{\|Tx\|}{\|x\|}\right),$$

and taking the supremum in the above over $\overline{\mathscr{D}(T)} \setminus \{0\}$ gives the desired second inequality. We thus conclude (2).
Lastly, The uniqueness of the extension follows from the fact that it is continuous on $\overline{\mathscr{D}(T)}$ and equals to T on $\mathscr{D}(T)$. Indeed, assume that S is a continuous extension of T and let $x \in \overline{\mathscr{D}(T)}$. There exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ that converges to x and we must have that

$$Sx = \lim_{n\to\infty} Sx_n \underset{S|_{\mathscr{D}(T)}=T}{=} \lim_{n\to\infty} Tx_n = \widetilde{T}x.$$

As x was arbitrary we have that $S = \widetilde{T}$.