Home Assignment 4

Exercise 1. Show that the multiplication operator $M : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ defined by

$$Mf(x) = m(x)f(x)$$

where $m \in C[a,b]$, satisfies

$$\|M\| = \|m\|_\infty .$$

Exercise 2. Prove the following statement: The integration operator $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ defined by

$$Tf(x) = \int_a^x f(t)dt.$$

satisfies

$$\|T\| = b - a.$$

Exercise 3. Prove the following statement: Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be normed spaces and let $T \in B(\mathcal{X}, \mathcal{Y})$ and $S \in B(\mathcal{Y}, \mathcal{Z})$. Then $S \circ T \in B(\mathcal{X}, \mathcal{Z})$ and

$$\|S \circ T\| \leq \|S\| \|T\| .$$

Consequently for any $T \in B(\mathcal{X}, \mathcal{X})$ the operator

$$T^n = \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}$$

is well defined, belongs to $B(\mathcal{X}, \mathcal{X})$, and satisfies

$$\left\| T^n \right\| \leq \|T\|^n .$$

Exercise 4. Let $\mathcal{X}$ be a normed space. Show that the function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \|x\|$ is not a linear functional (though it is continuous).

Exercise 5. Prove the following statement: Consider the Banach space $(C[a,b], \|\cdot\|_\infty)$. For any $x_0 \in [a,b]$ define the functional $\delta_{x_0} : C[a,b] \to \mathbb{F}$ by

$$\delta_{x_0}(f) = f(x_0).$$

Show that δ_{x_0} is in the dual space of $C[a,b]$.
δ_{x_0} is known as *the delta functional at* x_0. It appears in other contexts as well.

Exercise 6. Show that the map $f_\lambda : \ell_2(\mathbb{N}) \to \mathbb{C}$ defined by

$$f_\lambda(\boldsymbol{a}) = \sum_{n \in \mathbb{N}} n a_n \lambda^{n-1}$$

belongs to $\ell_2(\mathbb{N})^*$ for any $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and find $\boldsymbol{b}_\lambda \in \ell_2(\mathbb{N})$ such that

$$f_\lambda(\boldsymbol{a}) = \langle \boldsymbol{a}, \boldsymbol{b}_\lambda \rangle.$$

What is the norm of f_λ? Can we extend the above to the case where $|\lambda| = 1$?

Remark: You will need to start by showing that the sum which defines f_λ is finite.

Exercise 7. Prove the following statement: Let $\mathscr{H}$ be a Hilbert space. Define the map $\mathscr{I} : \mathscr{H} \to \mathscr{H}^*$ by

$$\mathscr{I} y = f_y$$

where $f_y(x) = \langle x, y \rangle$. Then $\mathscr{I}$ is a *conjugate linear* isometry between $\mathscr{H}$ and $\mathscr{H}^*$, i.e. $\mathscr{I}$ is a bijection such that for any $y_1, y_2 \in \mathscr{H}$ and a scalar α we have that

$$\mathscr{I}(y_1 + y_2) = \mathscr{I} y_1 + \mathscr{I} y_2, \qquad \mathscr{I}(\alpha y) = \overline{\alpha} \mathscr{I} y$$

and

$$\|\mathscr{I} y\|_{\mathscr{H}} = \|y\|_{\mathscr{H}}. \tag{1}$$

Consequently, we can define an inner product on $\mathscr{H}^*$ which induces the norm on the space by

$$\langle f, g \rangle = \overline{\langle \mathscr{I}^{-1} f, \mathscr{I}^{-1} g \rangle_{\mathscr{H}}}, \tag{2}$$

making $\mathscr{H}^*$ into a Hilbert space.

Exercise 8. We know from class for any $f \in \ell_1(\mathbb{N})^*$ there exists $\boldsymbol{b} \in \ell_\infty(\mathbb{N})$ such that $f = f_{\boldsymbol{b}}$ with $\|f_{\boldsymbol{b}}\| = \|\boldsymbol{b}\|_\infty$. Show that this implies that $\ell_1(\mathbb{N})^*$ is not separable.

Exercise 9. Prove the following statement: There exists a functional in $\ell_\infty(\mathbb{N})^*$ that is not of the form $f_{\boldsymbol{b}}$ with $\boldsymbol{b} \in \ell_1(\mathbb{N})$.

Hint: Show that

$$f_{\sum_{n=1}^N \alpha_n \boldsymbol{e}_n} = \sum_{n=1}^N \overline{\alpha_n} f_{\boldsymbol{e}_n}$$

and conclude that if every $\ell_\infty(\mathbb{N})^*$ is of the form $f_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{N})$ then $\ell_\infty(\mathbb{N})^*$ must be separable.

Exercise 10. Recall that we say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Banach space converges weakly to x, and write $x_n \xrightarrow[n\to\infty]{w} x$, if for any $f \in \mathscr{X}^*$ we have that

$$f(x_n) \underset{n\to\infty}{\longrightarrow} f(x).$$

Let $\mathscr{H}$ be a Hilbert space

(i) Show that $x_n \xrightarrow[n\to\infty]{w} x$ if and only if for any $y \in \mathscr{H}$

$$\langle x_n, y\rangle \underset{n\to\infty}{\longrightarrow} \langle x, y\rangle .$$

(ii) Let $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal set in $\mathscr{H}$. Show that $e_n \xrightarrow[n\to\infty]{w} 0$.

(iii) Does the statement in (ii) remains true if $\mathscr{B}$ is only orthogonal, i.e. if for any $x, y \in \mathscr{B}$ such that $x \neq y$ we have that $x \perp y$?

Exercise 11. In this exercise we'll investigate the uniqueness of weak limits.

(i) Let $\mathscr{H}$ be a Hilbert space. Show that the weak limit of any weak converging sequence is unique.

(ii) Using the fact (which we will prove in class) that for any x in a Banach space $\mathscr{X}$ there exists $f_x \in \mathscr{X}^*$ such that $f_x(x) = \|x\|$ to show that (i) remains true in general Banach spaces.

Exercise 12. Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a given sequence in $\mathscr{H}$.

(i) Show that if $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x then

$$\sup_{n\in\mathbb{N}} \|x_n\| < \infty, \tag{3}$$

and

$$\langle x_n, e_j\rangle \underset{n\to\infty}{\longrightarrow} \langle x, e_j\rangle \qquad \forall j \in \mathbb{N}. \tag{4}$$

In the remainder of this exercise we will aim to show that the converse to (i) holds.

(ii) Show that if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathscr{H}$ then for any $y \in \mathscr{H}$ we have that

$$\sum_{j=N+1}^{\infty} \left|\langle y, e_j\rangle\right| \left|\langle x_n, e_j\rangle\right| \leq M \sqrt{\sum_{j=N+1}^{\infty} \left|\langle y, e_j\rangle\right|^2}.$$

where $M = \sup_{n\in\mathbb{N}} \|x_n\|$.

(iii) Use the above to show that if (3) and (4) hold then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x.
Hint: Start by showing that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x if and only if $\{x_n - x\}_{n\in\mathbb{N}}$ converges weakly to 0, and continue by using Parseval's identity for inner products of Hilbert spaces with countable orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$:

$$\langle x, y\rangle = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle \overline{\langle y, e_n\rangle}.$$