Solution to Home Assignment 4

Solution to Question 1. For any $f \in C[a,b]$ we have that

$$\|Mf\| = \|mf\|_\infty = \max_{x\in[a,b]} |m(x)f(x)| \leq \|m\|_\infty \|f\|_\infty,$$

which implies that $\|M\| \leq \|m\|_\infty$.
Choosing $f \equiv 1$ we find that $\|f\|_\infty = 1$ and

$$\|M1\| = \|m\|_\infty.$$

Consequently

$$\|M\| = \sup_{\|f\|=1} \|Mf\|_\infty \geq \|M1\| = \|m\|_\infty.$$

Combining these two inequalities gives us the desired result.

Solution to Question 2. For any $f \in C[a,b]$ we have that

$$|Tf(x)| \leq \int_a^x |f(t)|\, dt \leq \int_a^x \|f\|_\infty\, dt = (x-a)\|f\|_\infty.$$

Consequently,

$$\|Tf\|_\infty \leq \max_{x\in[a,b]} \left((x-a)\|f\|_\infty\right) = (b-a)\|f\|_\infty,$$

which implies that $\|T\| \leq b-a$. As in the previous question, choosing $f \equiv= 1$ we find that $\|f\|_\infty = 1$ and

$$Tf(x) = x-a$$

which implies that $\|T\| \geq \|Tf\| = b-a$. Combining these two inequalities gives us the desired result.

Solution to Question 3. The linearity of this operator is a known result from Linear Algebra I and as such we won't show it and focus only on the boundedness. For any $x \in \mathscr{X}$ we have that

$$\|S \circ Tx\| = \|S(Tx)\| \leq \|S\| \|Tx\| \leq \|S\| (\|T\| \|x\|) = (\|S\| \|T\|) \|x\|.$$

This shows the boundedness of $S \circ T$ as well as the fact that

$$\|S \circ T\| \leq \|S\| \|T\|.$$

The second statement follows by induction. Indeed, it is a tautology for $n = 1$. Assume it holds for n and consider T^{n+1}. We have that

$$\|T^{n+1}\| = \|T \circ T^n\| \leq \|T\| \|T^n\| \leq \|T\| \|T\|^n = \|T\|^{n+1},$$

where we have used the first part of the question and the induction assumption.

Solution to Question 4. For any $\alpha < 0$ and any $x \neq 0$ we have that

$$\|\alpha x\| = |\alpha| \|x\| = -\alpha \|x\| \neq \alpha \|x\|.$$

This implies that the norm can't be a linear functional.

Solution to Question 5. For any $f, g \in C[a,b]$

$$\delta_{x_0}(f+g) = (f+g)(x_0) = f(x_0) + g(x_0) = \delta_{x_0}(f) + \delta_{x_0}(g).$$

Similarly, for any $f \in C[a,b]$ and a scalar α

$$\delta_{x_0}(\alpha f) = (\alpha f)(x_0) = \alpha f(x_0) = \alpha \delta_{x_0}(f),$$

which shows the linearity of δ_{x_0}. To show the boundedness of the functional we notice that

$$\left|\delta_{x_0}(f)\right| = \left|f(x_0)\right| \leq \|f\|_\infty$$

which implies that $\|\delta_{x_0}\| \leq 1$. Moreover, choosing $f \equiv 1$ we find that $\|f\|_\infty = 1$ and

$$\|\delta_{x_0}\| \geq \left|\delta_{x_0}(f)\right| = 1,$$

from which we can conclude that $\|\delta_{x_0}\| = 1$.

Solution to Question 6. We start by noticing that for any $x \in (-1.1)$ the function

$$g(x) = \sum_{n\in\mathbb{N}} x^{n-1} = \frac{1}{1-x}$$

is analytic and consequently $\sum_{n\in\mathbb{N}} n^m x^{k(n-1)}$ converges for any k and m in $\mathbb{N}$ when $|x| < 1$. We conclude that for any $|\lambda| < 1$

$$\sum_{n\in\mathbb{N}} n^2 |\lambda|^{2(n-1)} < \infty$$

which implies that the sequence $\boldsymbol{b}_\lambda$, defined by

$$b_{\lambda,n} = n\overline{\lambda}^{n-1}$$

belongs to $\ell_2(\mathbb{N})$. We notice that, by definition,

$$f_\lambda(\boldsymbol{a}) = \langle \boldsymbol{a}, \boldsymbol{b}_\lambda \rangle$$

and using Riesz' representation theorem we conclude that $f_\lambda \in \ell_2(\mathbb{N})^*$. The same theorem also tells us that

$$\|f_\lambda\| = \|\boldsymbol{b}_\lambda\|_2 = \sqrt{\sum_{n\in\mathbb{N}} n^2 |\lambda|^{2(n-1)}} = \frac{\sqrt{1+|\lambda|^2}}{\left(1-|\lambda|^2\right)^{\frac{3}{2}}}.$$

None of the above can be extended to when $|\lambda| = 1$. For example

$$f_1(\boldsymbol{a}) = \sum_{n\in\mathbb{N}} n a_n$$

which doesn't converge for all $\boldsymbol{a} \in \ell_2(\mathbb{N})$ (as the sequence $a_n = \frac{1}{n}$, which is in $\ell_2(\mathbb{N})$, illustrates).

Solution to Question 7. We start by showing that $\mathscr{I}$ is conjugate linear. Indeed, since for any $y_1, y_2 \in \mathscr{H}$ and any $x \in \mathscr{H}$

$$f_{y_1+y_2}(x) = \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle = f_{y_1}(x) + f_{y_2}(x) = \left(f_{y_1} + f_{y_2}\right)(x)$$

which implies that

$$\mathscr{I}\left(y_1 + y_2\right) = f_{y_1+y_2} = f_{y_1} + f_{y_2} = \mathscr{I} y_1 + \mathscr{I}_{y_2}.$$

Moreover, for any $y \in \mathscr{H}$ and a scalar α, and any $x \in \mathscr{H}$

$$f_{\alpha y}(x) = \langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle = \overline{\alpha} f_y(x)$$

which implies that

$$\mathscr{I}\left(\alpha y\right) = f_{\alpha y} = \overline{\alpha} f_y = \overline{\alpha} \mathscr{I} y.$$

Next we focus on the norm identity. Due to Riesz' representation theorem

$$\left\|\mathscr{I} y\right\|_{\mathscr{H}^*} = \sup_{x\in\mathscr{H},\, \|x\|=1} \left|\left(\mathscr{I} y\right)(x)\right| = \sup_{x\in\mathscr{H},\, \|x\|=1} \left|f_y(x)\right| = \left\|y\right\|_{\mathscr{H}}.$$

Riesz' representation theorem also assures us that $\mathscr{I}$ is surjective and the injectivity of it follows from the above. Indeed, if $\mathscr{I} y_1 = \mathscr{I} y_2$ then

$$0 = \left\|\mathscr{I} y_1 - \mathscr{I} y_2\right\|_{\mathscr{H}^*} = \left\|\mathscr{I}\left(y_1 - y_2\right)\right\|_{\mathscr{H}^*} = \left\|y_1 - y_2\right\|_{\mathscr{H}}$$

which implies that $y_1 = y_2$. The first part of the question is now concluded.

We now consider the function defined by (1) and show that it is an inner produce that induces $\|\cdot\|_{\mathscr{H}^*}$. Before we begin we mention that when a map $\mathscr{I}$ is conjugate-linear and bijective then its inverse, $\mathscr{I}^{-1}$ is also conjugate-linear and bijective (similar to an exercise in the previous assignment).

- For any $f \in \mathscr{H}^*$ we have that

$$\langle f, f \rangle = \overline{\langle \mathscr{I}^{-1} f, \mathscr{I}^{-1} f \rangle_{\mathscr{H}}} = \left\|\mathscr{I}^{-1} f\right\|_{\mathscr{H}}^2$$

  which shows the non-negativity. Moreover, $\langle f, f \rangle = 0$ if and only if $\mathscr{I}^{-1} f = 0$ which implies that[1]

$$f = \mathscr{I}\left(\mathscr{I}^{-1} f\right) = \mathscr{I} 0 = 0.$$

- For any $f, g, h \in \mathscr{H}^*$ we have that

$$\langle f + g, h \rangle = \overline{\langle \mathscr{I}^{-1}\left(f + g\right), \mathscr{I}^{-1} h \rangle_{\mathscr{H}}} = \overline{\langle \mathscr{I}^{-1} f + \mathscr{I}^{-1} g, \mathscr{I}^{-1} h \rangle_{\mathscr{H}}}$$
$$= \overline{\langle \mathscr{I}^{-1} f, \mathscr{I}^{-1} h \rangle_{\mathscr{H}}} + \overline{\langle \mathscr{I}^{-1} g, \mathscr{I}^{-1} h \rangle_{\mathscr{H}}} = \langle f, h \rangle + \langle g, h \rangle.$$

[1] this can be shown from the conjugate-linearity.

- For any $f, g \in \mathscr{H}$ and a scalar α we have that
$$\langle \alpha f, g\rangle = \overline{\langle \mathscr{I}^{-1}(\alpha f), \mathscr{I}^{-1} g\rangle_{\mathscr{H}}} = \overline{\langle \overline{\alpha}\mathscr{I}^{-1} f, \mathscr{I}^{-1} g\rangle_{\mathscr{H}}}$$
$$= \overline{\overline{\alpha}\langle \mathscr{I}^{-1} f, \mathscr{I}^{-1} g\rangle_{\mathscr{H}}} = \alpha\overline{\langle \mathscr{I}^{-1} f, \mathscr{I}^{-1} g\rangle_{\mathscr{H}}} = \alpha\langle f, g\rangle.$$
- For any $f, g \in \mathscr{H}$
$$\langle f, g\rangle = \overline{\langle \mathscr{I}^{-1} f, \mathscr{I}^{-1} g\rangle_{\mathscr{H}}} = \overline{\overline{\langle \mathscr{I}^{-1} g, \mathscr{I}^{-1} f\rangle_{\mathscr{H}}}}$$
$$= \langle \mathscr{I}^{-1} g, \mathscr{I}^{-1} f\rangle_{\mathscr{H}} = \overline{\langle g, f\rangle}.$$

As all the required properties are satisfied, we find that $\langle\cdot,\cdot\rangle$ is indeed an inner product in $\mathscr{H}^*$. To show that it induced the norm we need to show that $\|f\|^2_{\mathscr{H}^*} = \langle f, f\rangle$ for every $f \in \mathscr{H}^*$. Since, as we saw,
$$\langle f, f\rangle = \|\mathscr{I}^{-1} f\|^2_{\mathscr{H}}$$
it is enough to show that $\|f\|_{\mathscr{H}^*} = \|\mathscr{I}^{-1} f\|_{\mathscr{H}}$ for every $f \in \mathscr{H}^*$ which follows directly from (1). Indeed for any $f \in \mathscr{H}^*$
$$\|f\|_{\mathscr{H}^*} = \|\mathscr{I}(\mathscr{I}^{-1} f)\|_{\mathscr{H}^*} = \|\mathscr{I}^{-1} f\|_{\mathscr{H}}.$$

Solution to Question 8. In a previous assignment we have shown that $\ell_\infty(\mathbb{N})$ is not separable by finding an uncountable set $\{x_\alpha\}_{\alpha\in\mathscr{I}} \subset \ell_\infty(\mathbb{N})$ such that
$$\|x_\alpha - x_\beta\|_\infty \geq 1, \qquad \forall \alpha \neq \beta.$$
Consequently, the set $\{f_{x_\alpha}\}_{\alpha\in\mathscr{I}}$ is an uncountable set in $\ell_1(\mathbb{N})^*$ and
$$\left\|f_{x_\alpha} - f_{x_\beta}\right\| = \|x_\alpha - x_\beta\|_\infty \geq 1, \qquad \forall \alpha \neq \beta.$$
Thus, $\ell_1(\mathbb{N})^*$ is not separable.

Solution to Question 9. We have seen in class that if $f \in \ell_\infty(\mathbb{N})^*$ is of the form $f = f_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{N})$ then
$$(*) \qquad \|f_{\boldsymbol{b}}\| \leq \|\boldsymbol{b}\|_1.$$
Let $\mathscr{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ be the standard Scahuder basis of $\ell_1(\mathbb{N})$ and denote by $M = \{f_{\boldsymbol{e}_n}\}_{n\in\mathbb{N}}$. We claim that
$$f_{\sum_{n=1}^N \alpha_n \boldsymbol{e}_n} = \sum_{n=1}^N \overline{\alpha_n} f_{\boldsymbol{e}_n}$$
for any scalars $\alpha_1, \dots, \alpha_N$. Indeed, given any $\boldsymbol{a} \in \ell_\infty(\mathbb{N})$ we see that
$$f_{\sum_{n=1}^N \alpha_n \boldsymbol{e}_n}(\boldsymbol{a}) = \sum_{j\in\mathbb{N}} a_j \overline{\left(\sum_{n=1}^N \alpha_n \boldsymbol{e}_n\right)_j} = \sum_{j\in\mathbb{N}} a_j \overline{\sum_{n=1}^N \alpha_n \delta_{n,j}}$$

$$= \sum_{j=1}^{N} a_j \overline{\alpha_j} = \sum_{n=1}^{N} \overline{\alpha_j} f_{e_n}(\boldsymbol{a}).$$

Since $\boldsymbol{a} \in \ell_\infty(\mathbb{N})$ was arbitrary we conclude the desired identity. This, together with $(*)$ shows that for any $\boldsymbol{b} \in \ell_1(\mathbb{N})$ we have that

$$\left\| f_{\boldsymbol{b}} - \sum_{n=1}^{N} \overline{b_n} f_{e_n} \right\| = \left\| f_{\boldsymbol{b}} - f_{\sum_{n=1}^{N} b_n \boldsymbol{e}_n} \right\| \leq \left\| \boldsymbol{b} - \sum_{n=1}^{N} b_n \boldsymbol{e}_n \right\|_1 \underset{n\to\infty}{\longrightarrow} 0.$$

Consequently, if $\{f_{\boldsymbol{b}}\}_{\boldsymbol{b}\in\ell_1(\mathbb{N})} = \ell_\infty(\mathbb{N})^*$ we find that span$M$ is dense in $\ell_\infty(\mathbb{N})^*$. Since M is countable we conclude that $\ell_\infty(\mathbb{N})^*$ is separable. From class we know that $\mathcal{X}^*$ is separable implies that $\mathcal{X}$ is also separable and as we know that $\ell_\infty(\mathbb{N})$ is not separable we have reached a contradiction.

Solution to Question 10. (i) According to Reisz' representation theorem any $f \in \mathcal{H}^*$ can be written as f_y for some $y \in \mathcal{H}$ where

$$f_y(x) = \langle x, y \rangle$$

and for every $y \in \mathcal{H}$, $f_y \in \mathcal{H}^*$. Thus, $x_n \underset{n\to\infty}{\overset{w}{\longrightarrow}} x$ if and only if for every $y \in \mathcal{H}$

$$\langle x_n, y \rangle \underset{n\to\infty}{\longrightarrow} \langle x, y \rangle.$$

(ii) Since $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ is an orthonormal set we have that for any $y \in \mathcal{H}$

$$\sum_{n\in\mathbb{N}} \left| \langle y, e_n \rangle \right|^2 < \infty.$$

Thus, for any $y \in \mathcal{H}$ we must have that

$$\langle e_n, y \rangle = \overline{\langle y, e_n \rangle} \underset{n\to\infty}{\longrightarrow} 0 = \langle 0, y \rangle.$$

Using the previous sub-question we conclude the desired result.

(iii) The statement doesn't remain true. Consider the orthogonal sequence $x_n = ne_n$ where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence. Let

$$y = \sum_{n\in\mathbb{N}} \frac{1}{n} e_n.$$

Then y is well defined since $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$ and

$$\langle x_n, y \rangle = 1 \underset{n\to\infty}{\not\longrightarrow} 0$$

Solution to Question 11. (i) Let us assume that a given sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ converges weakly to x and y. Then, according to the previous exercise, for any $z \in \mathcal{H}$

$$\langle x_n, z \rangle \underset{n\to\infty}{\longrightarrow} \langle x, z \rangle$$

and

$$\langle x_n, z\rangle \underset{n\to\infty}{\longrightarrow} \langle y, z\rangle.$$

Consequently $\langle x, z\rangle = \langle y, z\rangle$, or $\langle x-y, z\rangle = 0$, for any $z \in \mathscr{H}$. Choosing $z = x-y$ we see that

$$\left\|x-y\right\|^2 = \langle x-y, x-y\rangle = 0$$

which shows that $x = y$, i.e. the weak limit is unique.

(ii) We use the given fact and follow up as in the previous sub-question: Assuming that $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ converges weakly to x and y we have that for any $f \in \mathscr{X}^*$

$$f\left(x-y\right) = f(x) - f(y) = \lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(x_n) = 0.$$

Choosing $f_{x-y} \in \mathscr{X}^*$ such that $f_{x-y}\left(x-y\right) = \left\|x-y\right\|$ in the above shows that $\left\|x-y\right\| = 0$ which implies that $x = y$.

Solution to Question 12. (i) (3) follows from a theorem form class and (4) follows from the previous question.

(ii) We have that

$$\sum_{j=N+1}^{\infty} \left|y, e_j\right| \left|\langle x_n, e_j\rangle\right| \leq \sqrt{\sum_{j=N+1}^{\infty} \left|y, e_j\right|^2} \sqrt{\sum_{j=N+1}^{\infty} \left|\langle x_n, e_j\rangle\right|^2} \leq \sqrt{\sum_{j=N+1}^{\infty} \left|y, e_j\right|^2} \left\|x_n\right\|.$$

where we have used the Cauchy-Schwarz and Bessel's inequalities. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded and

$$M = \sup_{n\in\mathbb{N}} \left\|x_n\right\|$$

we conclude that

$$\sum_{j=N+1}^{\infty} \left|\langle y, e_j\rangle\right| \left|\langle x_n, e_j\rangle\right| \leq M \sqrt{\sum_{j=N+1}^{\infty} \left|\langle y, e_j\rangle\right|^2}.$$

(iii) We start by showing the statement in the hint: $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x if and only if for any $y \in \mathscr{H}$

$$\langle x_n, y\rangle \underset{n\to\infty}{\longrightarrow} \langle x, y\rangle$$

which is equivalent to

$$\langle x_n, y\rangle - \langle x, y\rangle \underset{n\to\infty}{\longrightarrow} 0.$$

Due to the linearity of the inner product the above is equivalent to

$$\langle x_n - x, y\rangle \underset{n\to\infty}{\longrightarrow} 0 = \langle 0, y\rangle$$

for every $y \in \mathscr{H}$, which holds if and only if $\{x_n - x\}_{n\in\mathbb{N}}$ converges to 0. We thus focus our attention on using (3) and (4) to show that for any $y \in \mathscr{H}$ we have that

$$\left\langle x_n - x, y\right\rangle \underset{n\to\infty}{\longrightarrow} 0.$$

Given $y \in \mathscr{H}$ we have that

$$\left\langle x_n - x, y\right\rangle = \sum_{j\in\mathbb{N}} \left\langle x_n - x, e_j\right\rangle \overline{\left\langle y, e_j\right\rangle}.$$

For any given $N \in \mathbb{N}$ we find that

$$\left|\left\langle x_n - x, y\right\rangle\right| \leq \sum_{j=1}^{N} \left|\left\langle x_n - x, e_j\right\rangle\right| \left|\left\langle y, e_j\right\rangle\right| + \sum_{j=N+1}^{\infty} \left|\left\langle x_n - x, e_j\right\rangle\right| \left|\left\langle y, e_j\right\rangle\right|.$$

For a given $\varepsilon > 0$, since $\left\{\left\langle y, e_j\right\rangle\right\}_{j\in\mathbb{N}} \in \ell_2(\mathbb{N})$ we can find $N(\varepsilon) \in \mathbb{N}$ such that for any $N \geq N(\varepsilon)$

$$\sum_{j=N+1}^{\infty} \left|\left\langle y, e_j\right\rangle\right|^2 \leq \frac{\varepsilon^2}{M^2}$$

where $M = \sup_{n\in\mathbb{N}} \|x_n - x\|$ which is finite according to (3)[2]. Using the reult of our previous sub-question we conclude that for any $n \in \mathbb{N}$, as long as $N \geq N(\varepsilon)$ we have that

$$\sum_{j=N+1}^{\infty} \left|\left\langle x_n - x, e_j\right\rangle\right| \left|\left\langle y, e_j\right\rangle\right| \leq M \sqrt{\sum_{j=N+1}^{\infty} \left|\left\langle y, e_j\right\rangle\right|^2} < \varepsilon.$$

Since (3) is satisfied, for any given $\varepsilon > 0$ we can find $N(\varepsilon) \in \mathbb{N}$ such that for any $N \geq N(\varepsilon)$

$$\sum_{j=N+1}^{\infty} \left|\left\langle x_n - x, e_j\right\rangle\right| \left|\left\langle y, e_j\right\rangle\right| < \varepsilon.$$

Consequently, when $N \geq N(\varepsilon)$ we have that

$$\left|\left\langle x_n - x, y\right\rangle\right| \leq \sum_{j=1}^{N} \left|\left\langle x_n - x, e_j\right\rangle\right| \left|\left\langle y, e_j\right\rangle\right| + \varepsilon.$$

Using (4) we conclude that for any such N

$$\limsup_{n\to\infty} \left|\left\langle x_n - x, y\right\rangle\right| \leq \varepsilon.$$

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[2]

$$\sup_{n\in\mathbb{N}} \|x_n - x\| \leq \sup_{n\in\mathbb{N}} \left(\|x_n\| + \|x\|\right) = \sup_{n\in\mathbb{N}} \|x_n\| + \|x\|.$$

As ε was arbitrary and $\{|\langle x_n - x, y\rangle|\}_{n\in\mathbb{N}}$ is non-negative conclude that

$$\lim_{n\to\infty} \left|\langle x_n - x, y\rangle\right| = 0,$$

which shows the desired result.