Home Assignment 5

Exercise 1. Prove the Hahn-Banach theorem in the case where the underlying space is a Hilbert space.

Exercise 2. Use Baire's category theorem to show that any Hamel basis for an infinite dimensional Banach space must be uncountable.
Hint: Assume by contradiction that $\mathscr{B} = \{x_n\}_{n\in\mathbb{N}}$ *is a Hamel basis and consider the subspaces* $M_n = \text{span}\{x_1, \dots, x_n\}$.

Exercise 3. Prove the following statement: Let $\mathscr{X}$ be a Banach space and let $\mathscr{Y}$ be a normed space. If $\{T_n\}_{n\in\mathbb{N}} \in B\left(\mathscr{X}, \mathscr{Y}\right)$ satisfies that

$$\lim_{n\to\infty} T_n x \qquad \text{exists for all } x \in \mathscr{X}$$

then the map $T : \mathscr{X} \to \mathscr{Y}$ defined by

$$Tx = \lim_{n\in\mathbb{N}} T_n x$$

is a bounded linear map.

Exercise 4. Consider the vector subspace $\ell_c(\mathbb{N}) \subset \ell_\infty(\mathbb{N})$ defined by

$$\ell_c(\mathbb{N}) = \left\{\{a_n\}_{n\in\mathbb{N}} \in \mathbb{C} \mid \exists n_0 \in \mathbb{N} \text{ such that } a_n = 0 \ \ \forall n \geq n_0\right\}$$

and define the operators $T_n : \ell_c(\mathbb{N}) \to \ell_c(\mathbb{N})$ by

$$(T_n \boldsymbol{a})_k = \begin{cases} n a_n & k = n \\ 0 & k \neq n \end{cases}.$$

Show that for any $\boldsymbol{a} \in \ell_c(\mathbb{N})$ we have that $\sup_{n\in\mathbb{N}} \|T_n \boldsymbol{a}\| < \infty$ but

$$\sup_{n\in\mathbb{N}} \|T_n\| = \infty.$$

Why doesn't this violate the Banach-Steinhaus theorem?

The goal of the next exercise is to prove the following theorem:

Theorem *(Biorthogoanal sequence)*. Let $\mathscr{X}$ be a Banach space with Schauder basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$. Then there exists a unique sequence $\left\{f^{(n)}\right\}_{n\in\mathbb{N}} \in \mathscr{X}^*$ such that $f^{(n)}\left(e_j\right) = \delta_{n,j}$.

The proof will rely on the next theorem (which you can also do as an exercise, though a solution is not provided):

Theorem *(Sinai's theorem)*. Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space and let $\{e_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{X}$ such that $e_n \neq 0$ for every $n \in \mathbb{N}$. Define

$$\mathscr{Y} = \left\{\boldsymbol{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}} \in \mathbb{F} \mid \sum_{n\in\mathbb{N}} \alpha_n e_n \text{ converges in } \mathscr{X}\right\}$$

and set

$$\|\boldsymbol{\alpha}\|_{\mathscr{Y}} = \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \alpha_n e_n \right\|.$$

Then $\mathscr{Y}$ is a Banach space.

Exercise 5. Let $\mathscr{X}$ be a Banach space with Schauder basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$ and let $\mathscr{Y}$ be the space defined Sinai's theorem.

(i) Show that the map $T : \mathscr{Y} \to \mathscr{X}$ defined by

$$T\boldsymbol{\alpha} = \sum_{n\in\mathbb{N}} \alpha_n e_n$$

is a bounded linear operator that is a bijection between $\mathscr{Y}$ and $\mathscr{X}$.

(ii) Show that T^{-1} is bounded and conclude that there exists $C > 0$ such that for any $N \in \mathbb{N}$ and any $x \in \mathscr{X}$

$$\left\| \sum_{n=1}^{N} \alpha_n(x) e_n \right\| \le C \|x\|$$

where $x = \sum_{n\in\mathbb{N}} \alpha_n(x) e_n$ is the unique expansion of x with respect to the basis $\mathscr{B}$.

(iii) Prove the Biorthogonal sequence theorem by defining

$$f_n(x) = f_n\left(\sum_{j\in\mathbb{N}} \alpha_j(x) e_j \right) = \alpha_n(x)$$

and proving that f_n is linear with $\|f_n\| \le \frac{2C}{\|e_n\|}$.

Exercise 6. Let

$$D : C^1[0,1] \subset C[0,1] \to C[0,1]$$

be the derivative operator, where $C[0,1]$ is imbued with the norm $\|\cdot\|_\infty$ and $C^1[0,1]$ is the space of all continuously differentiable functions on $[0,1]$. Show that D is a closed linear operator.