# Solution to Home Assignment 5

**Solution to Question 1**. Let $\mathscr{Y}$ be a subspace of a Hilbert space $\mathscr{H}$ and let $f$ be a bounded linear functional on it. As we saw in class, we can always extend $f$ to $\overline{\mathscr{Y}}$ in a unique way that preserves its operator norm. Thus, we can assume without loss of generality that $\mathscr{Y}$ is closed. Since $\mathscr{H}$ is a Hilbert space and $\mathscr{Y}$ is a closed subspace in $\mathscr{H}$, $\mathscr{Y}$ by itself is a Hilbert space with respect to the induced inner product. Since $f \in \mathscr{Y}^*$ we know, according to the Riesz representation theorem, that here exists $y_f \in \mathscr{Y}$ such that $f : \mathscr{Y} \to \mathbb{F}$ must be of the form

$$f(x) = \langle x, y_f \rangle, \qquad x \in \mathscr{Y}.$$

Moreover, $\|f\| = \|y_f\|$. An extension to $\mathscr{H}$ is now readily available - define $\widetilde{f} : \mathscr{H} \to \mathbb{F}$ by

$$\widetilde{f}(x) = \langle x, y_f \rangle.$$

This is a bounded linear functional which extends $f$ and, just like in the proof of Riesz' representation theorem, we have that $\|\widetilde{f}\| = \|y_f\| = \|f\|$.

**Solution to Question 2**. We shall prove the statement by contradiction. Assume that there is a countable Hamel basis, $\mathscr{B} = \{x_n\}_{n\in\mathbb{N}}$, to a given infinite dimensional Banach space $\mathscr{X}$. Define

$$\mathscr{M}_n = \operatorname{span}\{x_1, \dots, x_n\}.$$

$\mathscr{M}_n$ is a finite dimensional subspace of $\mathscr{X}$ and as such a closed set. Moreover, since $\mathscr{B}$ is a Hamel basis we find that

$$\mathscr{X} = \cup_{n\in\mathbb{N}} \mathscr{M}_n.$$

Thus, according to Baire's category theorem, there must exists $n_0 \in \mathbb{N}$, $x_0 \in \mathscr{X}$ and $r > 0$ such that

$$x_0 + B_r(0) = B_r(x_0) \subset \mathscr{M}_{n_0}.$$

This, however, implies that $x_0 \in \mathscr{M}_{n_0}$ and consequently, as $\mathscr{M}_{n_0}$ is a subspace, that

$$B_r(0) = -x_0 + B_r(x_0) \subset -x_0 + \mathscr{M}_{n_0} = \mathscr{M}_{n_0}.$$

We conclude from the above that any $x \in \mathscr{X}$ such that $\|x\| < r$ must belong to $\mathscr{M}_{n_0}$. Since $\mathscr{M}_{n_0}$ is closed to scalar multiplication we have if $x \neq 0$ then

$$x = \frac{2}{r} \underbrace{\frac{r}{2}\frac{x}{\|x\|}}_{\in B_r(0) \subset M_{n_0}} \in \mathscr{M}_{n_0},$$

and since 0 also must be in $\mathscr{M}_{n_0}$ we conclude that $\mathscr{M}_{n_0} = \mathscr{X}$ which contradicts the fact that $\mathscr{X}$ is infinite dimensional.

**Solution to Question 3**. We have seen in class, when we showed that $B\left(\mathscr{X},\mathscr{Y}\right)$ is a Banach space whenever $\mathscr{Y}$ is, that if $\{T_n x\}_{n\in\mathbb{N}}$ converges pointwise for any $x\in\mathscr{X}$ then the operator $T:\mathscr{X}\to\mathscr{Y}$ defined by

$$Tx=\lim_{n\to\infty} T_n x$$

is a linear operator. Moreover, we saw that if $\sup_{n\in\mathbb{N}}\|T_n\|<\infty$ then $T$ is bounded with

$$\|T\|\leq\sup_{n\in\mathbb{N}}\|T_n\|.$$

Since $\lim_{n\to\infty} T_n x$ exists for any $x$ we find that

$$\sup_{n\in\mathbb{N}}\|T_n x\|<\infty \qquad \forall x\in\mathscr{X}$$

and according to the Banach-Steinhaus theorem, this implies that

$$\sup_{n\in\mathbb{N}}\|T_n\|<\infty.$$

This concludes the proof.

**Solution to Question 4**. Let $\boldsymbol{a}\in\ell_c(\mathbb{N})$. Then, there exists $n_0\in\mathbb{N}$ such that for any $n>n_0$ we have that $a_n=0$. Thus for any $n>n_0$

$$T_n\boldsymbol{a}=\boldsymbol{0}$$

and consequently for any fixed $\boldsymbol{a}\in\ell_c(\mathbb{N})$

$$\sup_{n\in\mathbb{N}}\|T_n\boldsymbol{a}\|=\sup_{n\leq n_0}\|T_n\boldsymbol{a}\|=\max_{n\leq n_0}\sup_{k\in\mathbb{N}}|(T_n\boldsymbol{a})_k|=\max_{n\leq n_0} n|a_n|<\infty.$$

However, denoting the "standard basis" in $\ell_\infty(\mathbb{N})$ by $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ we find that $\boldsymbol{e}_n\in\ell_c(\mathbb{N})$ for all $n\in\mathbb{N}$ and $T_n\boldsymbol{e}_n=n\boldsymbol{e}_n$. As $\|\boldsymbol{e}_n\|_\infty=1$ we conclude that

$$\|T_n\|\geq\|T_n\boldsymbol{e}_n\|_\infty=n$$

showing that $\sup_{n\in\mathbb{N}}\|T_n\|=\infty$. This doesn't violate the Banach-Steinhaus theorem since $\ell_c(\mathbb{N})$ is not a Banach space (you can check that it is not complete).

**Solution to Question 5**. (i) We start by noticing that from the definition of $\mathscr{Y}$ the operator $T$ is well defined. Clearly

$$T\left(\boldsymbol{\alpha}+\boldsymbol{\beta}\right)=\sum_{n\in\mathbb{N}}\left(\alpha_n+\beta_n\right)e_n=\sum_{n\in\mathbb{N}}\alpha_n e_n+\sum_{n\in\mathbb{N}}\beta_n e_n=T\boldsymbol{\alpha}+T\boldsymbol{\beta}$$

and

$$T(c\boldsymbol{\alpha})=\sum_{n\in\mathbb{N}}(c\alpha_n)e_n=c\sum_{n\in\mathbb{N}}\alpha_n e_n=cT\boldsymbol{\alpha}$$

which shows that $T$ is linear. To show that $T$ is bounded we notice that

$$\|T\boldsymbol{\alpha}\|=\left\|\sum_{n\in\mathbb{N}}\alpha_n e_n\right\|=\left\|\lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n e_n\right\|$$

$$= \lim_{N\to\infty} \left\| \sum_{n=1}^{N} \alpha_n e_n \right\| \leq \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \alpha_n e_n \right\| = \|\boldsymbol{\alpha}\|_{\mathscr{Y}}.$$

Next we show that $T$ is a bijection. Since $\mathscr{B}$ is a Schauder basis we know that for any $x \in \mathscr{X}$ there exists $\boldsymbol{\alpha}(x)$ such that $x = \sum_{n\in\mathbb{N}} \alpha_n(x) e_n$. Consequently, $\boldsymbol{\alpha}(x) \in \mathscr{Y}$ and $T(\boldsymbol{\alpha}(x)) = x$, showing that $T$ is surjective.

To show that $T$ is injective we notice that $T\boldsymbol{\alpha} = 0$ if and only if

$$\sum_{n\in\mathbb{N}} \alpha_n e_n = 0.$$

Due to the uniqueness of the expansion with respect to $\mathscr{B}$ we must have that $\alpha_n = 0$ for all $n \in \mathbb{N}$, or equivalently that $\boldsymbol{\alpha} = 0$. Thus $T$ is injective.

(ii) Using the open mapping for injective linear operators (note that $\mathscr{R}(T) = \mathscr{X}$ is a Banach space) we conclude that $T^{-1} : \mathscr{X} \to \mathscr{Y}$ is bounded and as such for any $x \in \mathscr{X}$

$$\sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \alpha_n(x) e_n \right\| = \|\boldsymbol{\alpha}(x)\|_{\mathscr{Y}} = \left\| T^{-1}x \right\|_{\mathscr{Y}} \leq \left\| T^{-1} \right\| \|x\|.$$

We achieved the desired inequality with $C = \left\| T^{-1} \right\|$.

(iii) Much like we've seen in class, the uniqueness of the coefficients in the expansion of $x$ with respect to $\mathscr{B}$ implies that $f_n$ are all linear funcitonals. To show that they are bounded we notice that for any $n \geq 2$

$$\left| f_n(x) \right| = |\alpha_n(x)| = \frac{\|\alpha_n(x) e_n\|}{\|e_n\|} = \frac{\left\| \sum_{j=1}^{n} \alpha_j(x) e_j - \sum_{j=1}^{n-1} \alpha_j(x) e_j \right\|}{\|e_n\|}$$

$$\leq \frac{\left\| \sum_{j=1}^{n} \alpha_j(x) e_j \right\| + \left\| \sum_{j=1}^{n-1} \alpha_j(x) e_j \right\|}{\|e_n\|} \leq \frac{2C}{\|e_n\|} \|x\|.$$

For $n = 1$ we have that

$$\left| f_1(x) \right| = \frac{\|\alpha_1(x) e_1\|}{\|e_1\|} \leq \frac{C}{\|e_1\|} \|x\| \leq \frac{2C}{\|e_1\|} \|x\|.$$

This gives the the required proof to the Biorthogonal sequence theorem.

**Solution to Question 6**. The linearity of $D$ is known from Analysis I. We will focus on its closedness by showing that if we have a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C^1[0,1]$ such that

$$f_n \underset{n\to\infty}{\overset{\|\cdot\|_\infty}{\longrightarrow}} f, \qquad f_n' = Df_n \underset{n\to\infty}{\overset{\|\cdot\|_\infty}{\longrightarrow}} g$$

then $f \in C^1[0,1]$ and $g = Df$. Since $\{f_n'\}_{n\in\mathbb{N}}$ are continuous functions that converge uniformly to a function $g$ we know that $g \in C[0,1]$ and for any $x \in [0,1]$

$$\int_0^x f_n'(y)dy \underset{n\to\infty}{\longrightarrow} \int_0^x g(y)dy.$$

Denoting by $G(x) = \int_0^x g(y)dy$ we find, due to Fundamental Theorem of Calculus, that $G \in C^1[0,1]$ with $DG = g$. Moreover since $\{f_n\}_{n\in\mathbb{N}}$ are continuous functions that converge uniformly to a function $f$ we know that $f \in C[0,1]$ and for any $x \in [0,1]$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(f_n(0) + \int_0^x f_n'(y)dy\right) = f(0) + G(x).$$

Since $G \in C^1[0,1]$ we have that $f \in C^1[0,1] = \mathcal{D}(D)$ and as the difference between them is a constant we have that

$$Df = Dg = g.$$

We conclude that $D$ is indeed a closed operator.